Mathematical Reviews

Published monthly by The American Mathematical Society

TABLE OF CONTENTS

Volume 22, Number 11A November 1961 Reviews 10884–11415

MATHEMATICAL REVIEWS

Published by

The American Mathematical Society, 190 Hope St., Providence 6, R.I.

Editorial Office

Mathematical Reviews, 190 Hope St., Providence 6, R.I.

Subscription: Price $50 per volume ($25 per volume to individual members of sponsoring societies). There will be two volumes in 1962. Checks should be made payable to Mathematical Reviews. Subscriptions should be addressed to the American Mathematical Society, 190 Hope St., Providence 6, R.I.

The preparation of the reviews appearing in this publication is made possible by support provided by a grant from the National Science Foundation. The publication was initiated with funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers or proprietors of the publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Mathematical Reviews is published in 1961 in twelve monthly issues, each in two parts, A and B; and a single index issue covering both parts. Reviews and pages are numbered consecutively with respect to the issue order 1A, 1B, . . . , 12A, 12B. When the letter A or B is prefixed to a review number, it indicates in which part the review appears.

In 1962 Mathematical Reviews will be published in two volumes; each volume will consist of six monthly issues with a Part A and Part B for each monthly issue.

Journal references in Mathematical Reviews are now given in the following form: J. Brobdingnag. Acad. Sci. (7) **4 (82)** (1952/53), no. 3, 17–42 (1954), where after the abbreviated title one has: (series number) volume number (volume number in first series if given) (nominal date), issue number if necessary, first page–last page (imprint date). In case only one date is given, this will usually be interpreted as the nominal date and printed immediately after the volume number (this is a change from past practice in Mathematical Reviews where a single date has been interpreted as the imprint date). If no volume number is given, the year will be used in its place. The symbol ★ precedes the title of a book or other non-periodical which is being reviewed as a whole.

References to reviews in Mathematical Reviews before volume 20 (1959) are by volume and page number, as MR 19, 532: from volume 20 on, by volume and review number, as MR **20** #4387. Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parantheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

Printed in Great Britain by William Clowes and Sons, Limited, London and Beccles

Mathematical Reviews

Vol. 22, No. 11A November, 1961 10884–11415

GENERAL

See also 11196, 11197, 11407.

10884:

Ulam, S. M. ★A collection of mathematical problems. Interscience Tracts in Pure and Applied Mathematics, no. 8. Interscience Publishers, New York-London, 1960. xiii+150 pp. $5.00.

This fascinating little book is not exactly a collection of problems. It is of very uneven character ranging from very specific problems to minor discourses suggesting methods of attack on vast new topics. It is written in a kind of "off the cuff" manner and very relevant references are occasionally missing, e.g., to H. Busemann [Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 27–31; Pacific J. Math. **3** (1953), 1–12; MR **10**, 395; **14**, 1115] in connection with the first problem on convex bodies on p. 38. This, however, does not detract from the value of the book, in which every mathematician would like to browse and which should certainly stimulate many younger mathematicians. The topics covered are those encountered by the author throughout his work and are dominated by the set-theoretic and combinatorial point of view. The book consists of eight chapters: Set theory (27 pp.); Algebraic problems (8); Metric spaces (15); Topological spaces (13); Topological groups (5); Some questions in analysis (19); Physical systems (31); Computing machines as a heuristic aid (30). *A. Dvoretzky* (Jerusalem)

10885:

Van Thiel, Arnoud. ★Analytical dialectic. Uitgeverij "Helmond", Helmond, 1960. 43 pp. (3 inserts)

The following remarks are typical: "An actual distinction represented by a cube-edge has four phases: first it begins, then it ceases to begin, then it begins to end and finally it ends. . . . While the cube stands for actual distinction, the octahedron stands for possible distinction. The rhombic dodecahedron therefore stands for the relation between actuality and possibility and thus for 'becoming'." The "3 inserts" are plates containing perspective sketches of regular and semi-regular polyhedra. *H. S. M. Coxeter* (Toronto)

10886:

Weyl, H.; Landau, E.; Riemann, B. ★Das Kontinuum, und andere Monographien. Chelsea Publishing Co., New York, 1960. v+83+vii+117+120+vi+48 pp. $6.00.

Reprinting of four books, bound as one: H. Weyl, *Das Kontinuum* [Verlag von Veit & Comp., Leipzig, 1918]; H. Weyl, *Mathematische Analyse des Raumproblems* [Springer, Berlin, 1923]; E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie* [2nd ed., Springer, Berlin, 1929]; B. Riemann, *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, edited with comments by H. Weyl [3rd ed., Springer, Berlin, 1923].

10887:

Выгодский, М. Я. [Vygodskiĭ, M. Ya.]. ★Справочник по высшей математике [Handbook of higher mathematics]. 5th ed. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961. 783 pp. 1.34 r.

Take one of the more voluminous traditional textbooks in analytic geometry and calculus, remove all problems and almost all proofs, add many details, topics, and heuristic discussions not usually included, insert brief biographical notes and longer historical accounts of some topics, organize in brief captioned sections, and index thoroughly. The result is an excellent handbook designed primarily to afford engineers both ready reference and a first introduction to its contents, but useful to any student or teacher of mathematics.

Though the theoretical level is low and the approach old-fashioned, the treatment is correct and thorough. For example, matrices are not mentioned, but linear systems are treated fairly completely by determinants. The differential is handled classically but carefully. Exceptions, conditions, domains, and counter-examples are noted. Proofs are replaced largely by plausible arguments or by special cases whose generalizations would yield proofs, but brief derivations are often given, and the omission of proofs is not hidden. In short, here is an avowed "cookbook" whose quality is higher than that of most textbooks. There should be more like it in all languages and at all levels. *Kenneth May* (Northfield, Minn.)

10888:

Hasse, Helmut. ★Proben mathematischer Forschung in allgemeinverständlicher Behandlung. 2te Aufl. Schriftenreihe zur Mathematik, Heft 1. Otto Salle Verlag, Frankfurt am Main-Hamburg, 1960. viii+103 pp. DM 6.80.

The first edition [1955] was reviewed in MR **17**, 445.

10889:

Kurepa, George. Some principles of mathematical education. Proc. Internat. Congress Math. 1958, pp. 567–572. Cambridge Univ. Press, New York, 1960.

Verf. tritt dafür ein, dass grundlegende mathematische Begriffe wie Funktion, Menge, Folge und ebenso logische Begriffe, so die aussagenlogischen Verknüpfungen (Implikation usw.) und Quantoren, so allgemein und so früh wie möglich auf den Schulen gelehrt werden. In dieser

Allgemeinheit wird diese Forderung keinen Einsprüchen begegnen. Wenn aber Verf. eine Verallgemeinerung grundlegender mathematischer Begriffe fordert (so des Begriffes der Funktion—wobei sich Verf. gegen die Beschränkung auf eindeutige Zuordnung wendet) so wird dem von vielen Fachleuten nicht zugestimmt. Hier zur Verteidigung seines Standpunktes anzugeben: Komplexe (soll heissen: holomorphe) Funktionen sind ja schon im allgemeinen mehrdeutig, ist ein Missverstehen oder Übersehen der grössten Leistung von Riemann.

H. Behnke (Münster)

10890:

Crowder, Norman A. ★The arithmetic of computers: An introduction to binary and octal mathematics. A TutorText. Doubleday & Company Inc., Garden City, N.Y., 1960. viii+472 pp. $3.95.

The rather pretentious title disguises the fact that the subject matter is sixth-grade arithmetic. From the Preface: "This is a TutorText—a book written by a new technique developed through recent advances in automatic teaching methods.... The reader's rate of progress through the course is determined only by his facility for choosing right answers instead of wrong ones. It is not recommended, however, that the book be read in one sitting, or even in two or three."

The reviewer feels intuitively that this last sentence is connected to some form of the Induction Principle but is unable to articulate an appropriate formulation.

A. J. Lohwater (Providence, R.I.)

HISTORY AND BIOGRAPHY

See also 10899, 11226, B11442, B11443.

10891:

Spoglianti, Maria. Sui libri aritmetici di Euclide. I, II, III, IV. Period. Mat. (4) **38** (1960), 175–186, 234–253, 265–292; **39** (1961), 17–36.

The author deals with the arithmetical books (VII, VIII, and IX) of Euclid's *Elements*. In particular she elucidates the logical structure of the theory of integers and their ratios developed in book VII, which constitutes the oldest known treatise on arithmetic. Following the method introduced by U. Cassini she gives the main propositions of Euclid both in a literal Italian translation and in modern mathematical notation, which she supplements if necessary by an enunciation of the implied postulates and of critical remarks. In the symbolic expression use is made of Peano's symbolism.

E. J. Dijksterhuis (Bilthoven)

10892:

Clagett, Marshall. ★The science of mechanics in the Middle Ages. Publications in Medieval Science, 4. The University of Wisconsin Press, Madison; Oxford University Press, London; 1959. xxix+711 pp. (9 plates) $8.00.

In his history of mechanics, first published in 1788, Lagrange wrote of Archimedes and Galileo that "the interval which separated these two great geniuses disappears in the history of mechanics". This view persists today, despite the historical researches of Duhem and others, perhaps because the working scientist, if he has historical curiosity, is reluctant to trust the judgment of historians. The book under review destroys any valid excuse for such continuing ignorance, for, along with *The medieval science of weights* [ed. E. A. Moody and M. Clagett, Univ. of Wis. Press, Madison, Wis., 1952], it lays before us in English translation all the major sources from which a juster view of mediaeval mechanics may be drawn. The authors represented are Jordanus de Nemore, Johannes de Muris, Albert of Saxony, Trivisano, Gerald of Brussels, Thomas Bradwardine, William Heytesbury, Richard Swineshead, John of Holland, John Dumbleton, Oresme, Giovanni di Casali, Jacobus de Sancto Martino, Blasius of Parma, Francischus de Ferraria, Franciscus de Marchia, John Buridan. For easy comparison there are also passages from Greek sources, Leonardo da Vinci, Galileo, Beeckman, and Copernicus. Since mediaeval writings are difficult for the unpractised, the author has provided copious notes and short descriptive and summary chapters. He considers not only works of discovery but also expositions, so as to determine the extent to which new views and results were spread and received.

The major divisions of the book concern statics, kinematics, and dynamics. The greatest discoveries were made in the second of these, for it is no exaggeration to say that in the years 1328–1350 four scholars at Merton College, namely, Bradwardine, Heytesbury, Swineshead, and Dumbleton, succeeded in grasping and formulating the concept of uniformly accelerated motion and in developing correctly all of its major properties, an achievement traditionally reputed as one of Galileo's greatest. Progress in dynamics was slower; mediaeval proposals of dynamical concepts and laws must be regarded as incorrect or partial, intermediate between classical and modern ideas. More important than any specific discovery is the unquestionable proof that the Middle Ages, far from being a period of servile commentary on Aristotle as it is still described in university textbooks today, was the source of the Western approach to physics as a quantitative science in which mathematics is the essential tool of the intellect.

Because of the source material included, this book will be essential in any reasonably complete library of the mathematical sciences and will remain a classic for many years.

C. Truesdell (Bologna)

10893:

Euler, Leonhard. ★Vollständige Anleitung zur Algebra. Unter Mitwirkung von Joh. Niessner in revidierter Fassung neu herausgegeben von Jos. E. Hofmann. Reclam-Verlag, Stuttgart, 1959. 571 pp. DM 18.50.

Euler's classical introduction to algebra, first published in German in Petersburg in 1770 (a Russian translation had already been printed two years earlier) and long out of print in its original language, has finally been made available again in slightly modernized German. The present very carefully prepared edition is augmented by an excellent historical introduction describing Euler's life in the scientific situation of his time; by a summary of the content with valuable comments on many details of Euler's exposition; and by an index containing also the modern equivalents of obsolete mathematical terms. Most of the notes of Natani's German edition (1883) have been

retained, but Lagrange's additions of more than 100 pages to the French edition of 1774 (prepared by Joh. III Bernoulli) are not given.

While the first part of the book might well be described as an Introduction to College Algebra (beginning with the basic mathematical operations and ending with logarithms and infinite series), it is the second part which is of special interest. It leads from a very complete discussion of the solutions of equations up to the fourth degree to the final section on indeterminate analysis, the high-light of the book and a treasure chest in itself. It was this set of problems to which Lagrange, a master in the subject as Euler himself, added his remarks almost 200 years ago—yet even today we do not possess general methods for their solution.

The book contains innumerable illustrative examples (occasionally such as $1-1+1-1+-\cdots=\frac{1}{2}$) and proceeds in a very leisurely pace which provides for an easy, enjoyable reading even for high-school students. The last English version which the reviewer was able to locate was printed in 1840; thus a new inexpensive edition (including Lagrange's additions) should be most welcome.

C. J. Scriba (Toronto)

10894:

Opial, Z. The rectification of a logarithmic spiral in the works of E. Torricelli. Wiadom. Mat. (2) **3**, 251–265 (1960). (Polish)

10895:

Vuillemin, J. ★La philosophie de l'algèbre de Lagrange (Réflexions sur le mémoire de 1770–1771). Université de Paris. Les Conférences du Palais de la Découverte, Série D, No. 71. Édition du Palais de la Découverte, Paris, 1960. 24 pp. 1.60 NF.

The author gives a very brief analysis of the long memoir by Lagrange, *Réflexions sur la résolution algébrique des équations* (1771) on the solution of algebraic equations, in which Lagrange analyzed the method which successfully solves the third-degree equation and then concluded that by the introduction and analysis of a resolvent equation one might be able to solve higher-degree equations. However, the author's purpose is not to explain Lagrange's ideas but to point out that Lagrange's memoir approaches the problem of solving equations in three different ways. First he uses the a posteriori method, that is, he proceeds from the reduced equation (the septic) to the resolvent. Then he employs the a priori method, that is, given the resolvent he finds the reduced equation. Thirdly, Lagrange in the fourth section of his memoir gives what he calls the metaphysics of algebra, and which the author calls the true a priori method, namely, to discover conditions under which one can determine the resolvent of a given equation. This analysis of Lagrange's memoir is then used to show that the philosopher J. G. Fichte in his *Doctrine of knowledge* followed the same procedures. [See H. Hoffding, *A history of modern philosophy, Vol. II*, pp. 144–157, Dover, New York, 1955.] The author's point is that philosophers can profit from such analogies.

M. Kline (New York)

10896:

Euclide. ★L'Optique et la Catoptrique. Oeuvres traduites pour la première fois du grec en français avec une introduction et des notes par Paul Ver Eecke. Librairie Scientifique et Technique Albert Blanchard, Paris, 1959. xlviii + 126 pp. 28 NF.

This book is a French translation of Heiberg's Greek text of the *Optics* of Euclid, Theon of Alexandria's recension of the same work, and the *Catoptrics*, generally attributed to Euclid.

The translations are prefaced by a valuable introduction in which the writer discusses the authorship of the two works; considers the definitions, postulates, and propositions of both; and compares Theon's version with the original. This introduction also points out those parts of the treatises that are in error. The last part of the introduction traces the transmission of the works into Europe and lists various translations.

Throughout the entire book are notes amplifying the introduction and the text. *E. B. Allen* (Troy, N.Y.)

10897:

Van Durme, M. ★Correspondance Mercatorienne. De Nederlandsche Boekhandel, Antwerp, 1959. 285 pp. 350 F.

This correspondence of the famous 16th century geographer and mapmaker Mercator lists more than 200 pieces covering the period from 1537 to 1596. In addition to the reproduction of all letters which are known, the contents of others still missing have been described as far as possible. Each piece is preceded by a short summary in French. Much additional information is to be found in the well-documented annotations. Introduction, bibliography, a list of correspondents, a chronological table, and a large alphabetical index fill more than sixty pages of this carefully prepared edition.

The contents of the correspondence ranges from family affairs over cartographical problems to religious disputes. Of special interest to the mathematician is a letter (no. 152) in which Mercator outlines a plan for studying mathematics and comments on his own education in this field. Works by Gemma Frisius, Joh. Voegelin, Oronce Finée are recommended together with the first six books of Euclid.

C. J. Scriba (Toronto)

10898:

van der Pol, Balthasar. ★Selected scientific papers. 2 vols. Edited by H. Bremmer and C. J. Bouwkamp, with an introduction by H. B. G. Casimir. North-Holland Publishing Co., Amsterdam, 1960. xv + 1339 pp. (1 plate) $18.50.

The scientific work of Balthasar van der Pol covered pure mathematics, applied mathematics, radio, and electrical engineering. Hence, these two volumes of his selected scientific papers must of necessity be of only limited value to the mathematician. However, van der Pol's interests were broad, and even in mathematics, his papers covered number theory, special functions, operational calculus and nonlinear differential equations. In this last field he was a pioneer. Van der Pol's equation still is a landmark and model for students of the field.

There are a number of papers which would be of interest to the mathematician, e.g., On the stability of the solutions of Mathieu's equation [1928] (with M. J. O. Strutt), Tchebycheff polynomials and their relation to circular functions, Bessel's functions and Lissajou figures [1933], On potential and wave functions in N dimensions

[1933], Application of the operational or symbolic calculus to the theory of prime numbers [1938], and numerous papers on the operational calculus and the Laplace transform. In the field of applied mathematics the work of van der Pol covered many phases of the problem of radio-wave propagation around the earth. The highly mathematical papers (with Bremmer) published in the Philosophical Magazine for 1937 still are basic in this field.

This publication is welcome not only as a tribute to van der Pol but because his papers appeared in a variety of journals and many of these are difficult to locate today. Though apparently photo-offset from the actual publications, the two volumes are clearly printed, and the majority of the papers are in English. A few are in French and German and only an occasional one in Dutch.

M. Kline (New York)

LOGIC AND FOUNDATIONS

See also B11825.

10899:

Bocheński, I. M. ★A history of formal logic. Translated and edited by Ivo Thomas. University of Notre Dame Press, Notre Dame, Ind., 1961. xxii + 567 pp. $20.00.

A collection of passages from logical writers from Plato to Gödel, with interstitial comments by the author, arranged to illustrate the thesis that logic has not progressed along a single line, but has developed independently in the West and in India, and in the West has had three periods of growth (the Greek, the medieval and the modern-mathematical) with intervening periods of decay and oblivion. Within each of the sections thus arising, the arrangement is partly chronological, partly according to topics and problems, so that it is clear at a glance that, for example, in each of the three Western periods logicians were worried about self-referential paradoxes, about the nature and varieties of implication and about the relation between propositional logic and term-logic.

The Greek period of growth subdivides into an earlier part dominated by Aristotle and a later by Chrysippus the Stoic, much as modern mathematical logic was dominated first by Boole and then by Frege (of the latest period, dominated by Gödel, the author gives only an introductory glimpse: extracts from the incompleteness proof). In both cases, we find a class- or term-logic developed first and a propositional logic later, but in the modern period we have a rich logic of relations and of multiple quantification, of which both the Greek and the intervening medieval epochs have only rudiments. There was much more of it in the medieval epoch, all the same, than the author suggests, and higher-order quantification, and quantification into intensional contexts, were handled in the Middle Ages with some skill and freedom, and with an awareness of the traps. While the author misses much of this, he has enough from the medieval writers to make it worth advising mathematicians to look at them—especially at their theories of reference ('supposition') and their treatment of antinomies—and not just at the more obviously relevant modern section.

There are hints that the Indians also handled higher-order quantifications with skill, and even had the essence of Frege's definition of number, but there is not enough given (or known) of Indian logic for us yet to disentangle its real subtleties from idealistic confusions.

The extent to which later Western periods built on earlier is left problematic, but the selections and comment make it obvious that medieval logic is vastly less merely imitative than was thought, and modern logicians (Peirce apart) have only begun to be aware of their precursors. The author has an introductory section on logical historiography; the subject is plainly growing very rapidly, and this collection of sources amounts to an 'interim report'. Much has been done in the field even since its original publication in 1956, and the translator adds to it a section on Abelard.

A. N. Prior (Manchester)

10900:

Stebbing, L. S. ★A modern introduction to logic. Harper Torchbooks/The Science Library. Harper & Brothers, New York, 1961. xviii + 525 pp. $2.75.

The author's *Logic* deservedly has retained its place in the classroom during the thirty-odd years since its first publication [Methuen, London, 1931; book under review is reprinting of 7th ed., Humanities Press, New York, 1950]. Its approach constitutes a kind of amalgam between the traditional topics of formal logic (theory of the syllogism, relations between propositions, etc.), modern generalisation of logic, with brief introductory discussion of the symbolism of symbolic logic; and finally philosophical problems inherited from the writings of Moore, Russell, Whitehead and Broad, including extensive discussions of the theory of descriptions and the notion of existence. Part II treats of topics nowadays classed under the heading "Philosophy of science". Here the freshness of the chapters on the nature of scientific inquiry and theories, as well as those on causality and hypothesis, will continue to charm the attention of the student-reader; and the chapter on the "problem of induction" is still one of the best classical summaries of its time. There is a useful section on the historical development of logic; and if the book lacks some of the topics of more "up-to-date" texts of this kind, e.g., discussions on the logical status of laws, the problems of "concept-formation", operationalism, not to mention any detailed treatment of the social and historical sciences, the gap is compensated for by a certain freshness and excitement carried down from the "revolutionary years" of the Cambridge of the twenties.

G. Buchdahl (Cambridge, England)

10901:

Чёрч, А. [Church, Alonzo]. ★Введение в математическую логику. I [Introduction to mathematical logic. Vol. I]. Translated from the English by V. S. Černyavskiĭ; edited by V. A. Uspenskiĭ. Izdat. Inostr. Lit., Moscow, 1960. 484 pp. 23.40 r.

A translation into Russian of Vol. I of Alonzo Church's *Introduction to mathematical logic* [Princeton Univ. Press, Princeton, N.J., 1956; MR **18**, 631].

10902:

Cuzzer, Otto. Appunti sulla Logica e la Logistica. Period. Mat. (4) **36** (1958), 221–238.

L'auteur ne croit pas que la logique symbolique ait une valeur constructive; il affirme que le progrès réel des

sciences, de Galilée à Einstein, s'est accompli en se fondant sur l' "intuition scientifique" et sur la logique classique. L'auteur cite les critiques de H. Poincaré au symbolisme péanien; en suivant les idées de F. Enriques (les principes logiques expriment certaines "conditions d'invariance" dans notre expérience), il conclut en affirmant l'impossibilité de réduire la logique à un calcul, à un "mécanisme". *L. Lombardo-Radice* (Rome)

10903:

Castañeda, Hector Neri. Outline of a theory on the general logical structure of the language of action. Theoria (Lund) **26** (1960), 151–182.

The author outlines a theory of 'varieties of prescriptive discourse', such as deciding, commanding and permitting. He sets out a formalisation of prescriptive logic and discusses the semantics and functions of prescriptive discourse. *A. Rose* (Nottingham)

10904:

Spector, C. Hyperarithmetical quantifiers. Fund. Math. **48** (1959/60), 313–320.

The author addresses himself to a question raised in Kleene [Compositio Math. **14** (1959), 23–40; MR **21** #2586] and uses techniques developed in Kleene [Amer. J. Math. **66** (1944), 41–58; **77** (1955), 405–428; MR **5**, 197; **17**, 5]. To the definition of Q in (33) of the first of the latter pair of papers, the author adds the clause "$C(a)$ is linearly ordered by $x \in C(y)$ relative to $=$", and uses the primitive recursive predicate R_0 obtained from the definition to obtain the class $Q_0 = \hat{a}(x)(Ey)R_0(a, x, y)$. Where Kleene's set 0 is the smallest set satisfying the definition, Q_0 is the largest such set.

The author defines in terms of Q_0 a class Q_a for each natural number a and a binary predicate L. In terms of these, the main theorem is as follows. Let β be a variable which ranges over all two-place number-theoretic predicates, and let $a = 3.5^{(a)_2}$. Then $a \in 0 = (\mathrm{E}\beta)[\beta GHA \wedge L(a, \beta)] = (\mathrm{E}\beta)L(a, \beta)$. The question raised by Kleene is whether, given a recursive P, there is a primitive recursive R such that $(\mathrm{E}\alpha)[\alpha \in HA \wedge (x)R(\alpha, x, a)] \equiv (\alpha)(\mathrm{E}x)P(\alpha, x, a)$, where HA is the class of hyperarithmetical predicates. This question is answered affirmatively by the first corollary of the paper.

E. J. Cogan (Bronxville, N.Y.)

10905:

Shapiro, H. S. Numbers and functions computable by means of rational recurrence formulae. Comm. Pure Appl. Math. **12** (1959), 513–522.

A subclass of the primitive recursive functions is defined as follows: If P_i $(i = 1, \cdots, k)$ is a polynomial with integer coefficients, and f is one of a set of functions f_i satisfying $f_i(n+1) = P_i(f_1(n), \cdots, f_k(n))$, then f is P-recursive. The order of f is the least k for which such a system exists. (Some properties of P-recursive functions suggest a relationship with functions computable by finite automata.) A real number is R-recursive if it is the limit of a ratio $f(n)/g(n)$, where f and g are P-recursive. No assumptions are made as to the computability of the rate of convergence. Thus among a number of questions left open by the author is whether or not every R-recursive real number is recursive. The main part of the paper is devoted to establishing that certain operations (e.g., taking zeros of a polynomial) do not lead out of the class of R-recursive real numbers, and that such numbers as e and π are R-recursive. *H. G. Rice* (Santa Monica, Calif.)

10906:

Трахтенброт, Б. А. [Trahtenbrot, B. A.]. ★Алгоритмы и машинное решение задач [Algorithms and machine solution of problems]. 2nd ed.; edited by S. V. Yablonskiĭ. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. 119 pp. 1.85 r.

A popular exposition of recursive function theory with the following section headings: numerical algorithms, game algorithms, algorithms for searching a path in a labyrinth, word problem, computing machine with automatic control, program (machine algorithm), necessity of defining more precisely the concept of algorithm, Türing machine, realization of an algorithm on a Türing machine, basic hypothesis of the theory of algorithms, universal Türing machine, algorithmically unsolvable problems, impossibility of an algorithm for the problem of equivalence of words. *Walter Gautschi* (Oak Ridge, Tenn.)

10907:

Putnam, Hilary; Smullyan, Raymond M. Exact separation of recursively enumerable sets within theories. Proc. Amer. Math. Soc. **11** (1960), 574–577.

Let T be a recursively axiomatized theory in standard formalization. Two sets A and B are said to be separated by the formula $F(x)$, if $A \subseteq \{n : \vdash F(\bar{n})\}$ and $B \subseteq \{n : \vdash \neg F(\bar{n})\}$, where $\bar{n}$ denotes the nth numeral and $\vdash$ means 'is provable in T'. A and B are exactly separated by F if we have '$=$' instead of '$\subseteq$'. The authors show that if (i) T is consistent, (ii) all recursive functions are definable in T and (iii) every pair of disjoint recursively enumerable (r.e.) sets is separable in T, then every such pair of sets is exactly separable in T. This sharpens the main result of Ehrenfeucht and Feferman [Arch. Math. Logik Grundlagenforsch. **5** (1960), 37–41]. An alternative proof of the authors' result by means of a direct diagonal construction was given by Shepherdson [ibid., 119–127].

If T satisfies (i) and (ii), then (iii) is satisfied provided there is a formula $a \leqq b$ such that, for each n, (a) $\vdash x \leqq \bar{n} \leftrightarrow (x = 0 \vee x = 1 \vee \cdots \vee x = \bar{n})$ and (b) $\vdash x \leqq \bar{n} \vee \bar{n} \leqq x$, i.e., in each model $\mathfrak{M}$ of T the 'natural numbers' of $\mathfrak{M}$ (the denotations of $0, 1, \cdots, \bar{n}, \cdots$) precede all the other elements. Observe that the notion of separability has a model-theoretically interesting interpretation. Let $F_{\mathfrak{M}}(\bar{n})$ mean that n satisfies $F(x)$ when the quantifiers in F range over the model $\mathfrak{M}$ (of T). For any recursively axiomatized T and any pair A, B of r.e. non-recursive sets, there is a model in which, for all formulae G, $A \neq \{n : G_{\mathfrak{M}}(\bar{n})\}$ and $B \neq \{n : G_{\mathfrak{M}}(\bar{n})\}$. Nevertheless, under (iii), there exists an F such that in each model $\mathfrak{M}$ of T, $\{n : F_{\mathfrak{M}}(\bar{n})\}$ separates A and B. Exact separation does not seem to have a similarly clear model-theoretic meaning.

G. Kreisel (Paris)

10908:

Rabin, Michael O. On recursively enumerable and arithmetic models of set theory. J. Symb. Logic **23** (1958), 408–416.

Let Σ be the conjunction of the axioms of Gödel's set theory formulated in first-order logic with identity and with $\in$ as the only non-logical constant. The author proves that the sentence Σ has no recursively enumerable model. That is to say, if $I=\{0, 1, 2, \cdots\}$ is the set of non-negative integers and $E\subseteq I\times I$ is a binary relation such that the system $\langle I, E\rangle$ is a model of Σ, then the relation E is not recursively enumerable. Further, if E is in addition assumed to be an arithmetically definable relation, then it is shown that the system of "integers" of the "set theory" $\langle I, E\rangle$ is not isomorphic to the standard system of integers; indeed, there will be a true sentence of arithmetic whose set-theoretical version is false in the model $\langle I, E\rangle$. *Dana Scott* (Berkeley, Calif.)

10909:

Mal'cev, A. I. Model correspondences. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 313–336. (Russian)

A correspondence between two classes K and L of models is a relation $\mathfrak{M}\sigma\mathfrak{N}$ between models $\mathfrak{M}\in K$ and $\mathfrak{N}\in L$ such that if $\mathfrak{M}_1$ is isomorphic with $\mathfrak{M}$, and $\mathfrak{N}_1$ with $\mathfrak{N}$, then $\mathfrak{M}\sigma\mathfrak{N}\leftrightarrow\mathfrak{M}_1\sigma\mathfrak{N}_1$. Let the basic sets M_α of a model in K be indexed by $\alpha\in A$, and let the predicates P_γ of K be indexed by $\gamma\in\Gamma$ (similarly, for L, use N_β ($\beta\in B$) and Q_δ ($\delta\in\Delta$)). An axiomatic correspondence σ between K and L is given by a set of new predicates $S_\lambda^{i_1,\cdots,i_k}$ ($i_1, \cdots, i_k\in A\cup B$; $\lambda\in\Lambda$) and a set $\mathfrak{S}$ of first-order axioms (formulated in terms of the P_γ, Q_δ, and S_λ) such that, for $\mathfrak{M}\in K$ and $\mathfrak{N}\in L$, $\mathfrak{M}\sigma\mathfrak{N}$ if and only if one can define predicates S_λ such that $\mathfrak{S}$ holds for the model set $\{M_\alpha, N_\beta\}$. A correspondence σ is projective if and only if there is a new set C of indices and predicate symbols $S_\lambda^{i_1,\cdots,i_k}$ ($i_1, \cdots, i_k\in A\cup B\cup C$; $\lambda\in\Lambda$) and a set $\mathfrak{S}$ of first-order axioms (written in terms of the P_γ, Q_δ, S_λ) such that $\mathfrak{M}\sigma\mathfrak{N}$ if and only if there are non-empty sets T_μ ($\mu\in C$) and predicates S_λ such that $\mathfrak{S}$ holds. (These definitions can be extended to correspondences of more than two arguments.) A subclass K' of a class K of models is a projective subclass if and only if there is an axiomatic class L and a projective correspondence σ between K and L of which K' is the domain (i.e., $K'=\{\mathfrak{M}|\mathfrak{M}\in K$ & $(\exists\mathfrak{N})(\mathfrak{N}\in L$ & $\mathfrak{M}\sigma\mathfrak{N})\}$). (In case K is the class of all models of a given type, K' is called a projective class.) (1) Let σ be a projective correspondence among models of projective classes $K_1, \cdots, K_s$, and let $\mathfrak{M}_1, \cdots, \mathfrak{M}_s$ belong to $K_1, \cdots, K_s$ respectively. If every finite partial description $\mathfrak{M}_1', \cdots, \mathfrak{M}_s'$ can be imbedded in models $\mathfrak{N}_1', \cdots, \mathfrak{N}_s'$ which are in the relation σ, then $\mathfrak{M}_1, \cdots, \mathfrak{M}_s$ can be imbedded in models $\mathfrak{N}_1, \cdots, \mathfrak{N}_s$ in the relation σ. (Corollary: Assume that a projective class K is a subclass of an axiomatic class K_0, and any K_0-submodel of a K-model is a K-model. Then K is universally axiomatizable in K_0.) (2) (Boundedness) For each projective correspondence σ between projective classes $K_1, \cdots, K_s$, there is an infinite cardinal $\mathfrak{m}$ such that, if $\mathfrak{M}_1\in K_1, \cdots, \mathfrak{M}_s\in K_s$, $\sigma(\mathfrak{M}_1, \cdots, \mathfrak{M}_s)$ and $\mathfrak{U}_1, \cdots, \mathfrak{U}_s$ are subsets of $\mathfrak{M}_1, \cdots, \mathfrak{M}_s$ of power not exceeding some cardinal $\mathfrak{n}\geqq\mathfrak{m}$, then there are submodels $\mathfrak{N}_i$ of $\mathfrak{M}_i$ ($i=1, \cdots, s$) such that $\mathfrak{N}_i\in K_i$ $\sigma(\mathfrak{N}_1, \cdots, \mathfrak{N}_s)$, $\overline{\overline{\mathfrak{N}}}_i\leqq\mathfrak{n}$, and $\mathfrak{U}_i\subseteq\mathfrak{N}_i$. Also, for any infinite models $\mathfrak{M}_i\in K_i$ with $\sigma(\mathfrak{M}_1, \cdots, \mathfrak{M}_s)$, there exist $\mathfrak{N}_i\in K_i$ with $\sigma(\mathfrak{N}_1, \cdots, \mathfrak{N}_s)$, $\overline{\overline{\mathfrak{N}}}_i=\mathfrak{m}$, and $\mathfrak{M}_i\subset\mathfrak{N}_i$ ($\mathfrak{M}_i\neq\mathfrak{N}_i$) if $\overline{\overline{\mathfrak{M}}}_i\ngeqq\mathfrak{m}$. After the notion of quasi-universal class is introduced (too complex to describe here), the author proves that if a model $\mathfrak{M}$ has a local system of submodels belonging to a quasi-universal class L, then $\mathfrak{M}\in L$. Also, certain kinds of quasi-universal classes are shown to be universally axiomatizable. Applications of some of these results are made to RN, RI and Z groups.

E. Mendelson (New York)

10910:

Ohashi, Kenhachirō. On undecidable theorems. Sûgaku **9** (1957/58), 96–97. (Japanese)

In a consistent formal system Σ including recursive number theory, let the Gödel number of A be denoted by $\{A\}$, and let $\mathfrak{B}(\{A\})$ denote the formula in Σ expressing the fact that A is provable. The author proves that $\vdash\mathfrak{B}(\{A\})\supset A$ implies $\vdash A$ and that $\mathfrak{B}(\{A\})$ is undecidable for any A satisfying $\vdash\neg A$, assuming the following five conditions:

$$\vdash\mathfrak{B}(\{A\})\supset\mathfrak{B}(\{\mathfrak{B}(\{A\})\}),\ \vdash\mathfrak{B}(\{A\supset B\})\supset(\mathfrak{B}(\{A\})\supset\mathfrak{B}(\{B\})),$$

that $\vdash A$ implies $\vdash\mathfrak{B}(\{A\})$, that $\vdash\neg A$ does not hold if $\vdash\mathfrak{B}(\{A\})$, and that $\vdash\neg A$ implies $\vdash A\supset B$.

K. Ono (Nagoya)

10911:

Heyting, A. Axioms for intuitionistic plane affine geometry. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 160–173. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

The author wished to choose a problem about axiomatic theories which was "trivial in classical mathematics, so that the intuitionistic difficulties appear, so to say, in their purest form". The problem chosen centers around the well-known fact that an affine plane can be extended to a projective plane by the adjunction of points at infinity. The intuitionistic difficulties appear at once: given two lines we do not always know whether they are parallel or not, and hence we do not know whether their intersection should be at infinity or not. This difficulty is overcome by defining the notion of a projective point so that the decision of whether it is at infinity or not does not have to be made in advance. The verification of the projective axioms in terms of this construction was a bit disappointing to the reviewer: two special axioms had to be adjoined to those for affine geometry solely for the purpose of proving the usual projective axioms of incidence. Of course these special axioms are true in the analytical model of affine geometry based on the intuitionistic theory of real numbers, but still they seemed a little artificial.

Dana Scott (Berkeley, Calif.)

10912:

Ramsey, Frank Plumpton. ★The foundations of mathematics and other logical essays. Edited by R. B. Braithwaite with a preface by G. E. Moore. International Library of Psychology, Philosophy and Scientific Method. Littlefield, Adams & Co., Paterson, N.J., 1960. xviii+292 pp. $1.95.

Ramsey belonged to the Augustan Age of 20th-century Cambridge philosophy. He belonged to that small group of thinkers—including Russell, Moore, Whitehead, Wittgenstein—who were to change so decisively the whole face

of philosophy, and he might have been even more influential but for his untimely death. As it is, all we possess are a few precious fragments: a trenchant critique of *Principia Mathematica*, under the title "The foundations of mathematics" [1925], together with two minor papers on mathematical logic; a small series of philosophical papers dealing with such questions as the nature of universals, of facts and propositions, of truth and probability; and finally, the famous "Last papers" [1929] which include discussions of the logical structure of scientific theories, of laws of nature, causal qualities, and philosophy in general.

They all have influenced their generation, and republication, after thirty years, has diminished none of the sheer intellectual power one senses when coming into contact with Ramsey's mind. His views on theories have influenced many writers, including R. B. Braithwaite, the editor of this volume. Ramsey's analysis of the concept of scientific law (whilst being in line with the logico-positivists' doctrine that a law is not a proposition but a rule for the formation of propositions) is more subtly argued than theirs. At the same time we find Ramsey edging away from his earlier adherence to the ideas of Wittgenstein's *Tractatus*, that "a general proposition is equivalent to a conjunction of its instances" (p. 77), through the consideration that such a theory of conjunctions cannot be expressed for lack of symbolic power, adding the quip which has become classical: "But what we can't say we can't say, and we can't whistle it either" (p. 238). Readers of this journal will not fail to see the connection with Brouwer and the finitists!

The "Foundations of mathematics" is largely an attempt to remove three defects of the *Principia Mathematica*: (1) the neglect of the possibility of indefinable classes and relations in extension; (2) the axiom of reducibility; (3) a treatment of the concept of identity which makes the identity of indiscernibles analytic. Ramsey's theory of predicative functions was intended to dispose of the usual paradoxes by the use of an improved theory of types which made the employment of the axiom of reducibility unnecessary, and involving a complete extensionalising of mathematics. Among interesting sidelights, here, are Ramsey's arguments against Hilbert's formalism and Brouwer's intuitionism, though four years later, at the end of his life, he was converted to a finitist view which rejected the existence of any actual infinite aggregate.

G. Buchdahl (Cambridge, England)

10913:

Henkin, Leon. On mathematical induction. Amer. Math. Monthly **67** (1960), 323–338.

Verf. benutzt das Wort "Modell", um ein System $\mathscr{N}=\langle N, 0, S\rangle$, bestehend aus einer Menge N, einem Element 0 von N und einer (eindeutigen) einstelligen Operation S, die N in (nicht notwendig auf) N abbildet, damit zu bezeichnen. Ein Modell $\mathscr{N}=\langle N, 0, S\rangle$ heisst ein Peano-Modell, wenn es die folgenden (Peanoschen) Axiome befriedigt: (P1) Für alle $x\in N$, $Sx\neq 0$; (P2) für alle $x, y\in N$, wenn $x\neq y$, dann $Sx\neq Sy$; (P3) wenn G eine beliebige Teilmenge von N ist derart, dass (a) $0\in G$, und (b) immer wenn $x\in G$, auch $Sx\in G$, dann gilt $G=N$. Ferner nennt Verf. ein Modell ein Induktionsmodell, wenn es das Axiom (P3) (das Induktionsaxiom) befriedigt.

Der, für die Axiomatik der natürlichen Zahlen recht bedeutsame, Aufsatz hat zum Hauptinhalt, dass eine notwendige und hinreichende Bedingung dafür, dass in einem Modell alle möglichen Definitionen durch mathematische Induktion gerechtfertigt sind, ist, dass dieses Modell ein Peano-Modell ist. Doch können natürlich gewisse Definitionen durch mathematische Induktion auch in Induktionsmodellen, die keine Peano-Modelle sind, möglich sein, so die Definitionen durch mathematische Induktion der (einzigen) Operationen der Addition (Satz II) und der Multiplikation. Doch existiert z.B. nicht die Operation der Exponentiation in jedem Induktionsmodell.

Der Beweis der obigen Tatsachen wird mittels der folgenden Sätze erbracht. Eine notwendige und hinreichende Bedingung dafür, dass für beliebige Modelle $\mathscr{N}$ und $\mathscr{N}_1=\langle N_1, 0_1, S_1\rangle$ ein einziger Homomorphismus von $\mathscr{N}$ in $\mathscr{N}_1$ existiert, ist, dass $\mathscr{N}$ ein Peano-Modell ist (Satz I und VIII). Eine notwendige und hinreichende Bedingung dafür, dass ein beliebiges Modell $\mathscr{N}_1$ ein homomorphes Abbild eines Peano-Modells $\mathscr{N}$ ist, ist dass $\mathscr{N}_1$ ein Induktionsmodell ist (Satz IV). Dabei heisst ein Modell $\mathscr{N}_1$ ein homomorphes Abbild des Modells $\mathscr{N}$, wenn ein Homomorphismus h von $\mathscr{N}$ auf $\mathscr{N}_1$ existiert. Sei $\mathscr{N}=\langle N, 0, S, +\rangle$ ein ("erweitertes") Induktionsmodell. Dann existiert ein einziges Homomorphismus h von $\mathscr{N}^*=\langle N^*, 0^*, S^*, +^*\rangle$ auf $\mathscr{N}$, d.h. (auch) die Operation $+$ (bezüglich $\mathscr{N}$) ist das Abbild (unter dem Homomorphismus h) der Operation $+^*$ (bezüglich $\mathscr{N}^*$). Dabei kann das System $\langle N^*, 0^*, S^*, +^*\rangle$ das Peano-Modell der natürlichen Zahlen bedeuten, so wie wir es intuitiv kennen (Satz VI). Ein entsprechender Satz gilt auch für die Multiplikation.

Der früher erschienene Aufsatz von Hanfried Lenz [Acta Math. Acad. Sci. Hungar. **9** (1958), 33–44; MR **20** #2284] verfolgt dasselbe Ziel wie der vorliegende Aufsatz, nämlich eine solche Einführung der elementaren Rechnenoperationen zu bewerkstelligen, die dem Standpunkt der neueren Algebra am bestehen gerecht wird. Lenz bedient sich dabei sogar derselben Methode der Heranziehung der oben beschriebenen Homomorphismen. Die Lenzsche Resultaten sind ausserdem teilweise allgemeiner, da sich dieselben auf beliebige Zahlensysteme beziehen, so auf das System der ganzen Zahlen. Lenz erreicht das dadurch, dass er einen Kunstgriff benutzt und das Induktionsaxiom in einem gewissen Sinne abschwächt, so dass nunmehr alle Systeme umfasst werden, die den Namen "Zahlensysteme" verdienen, während das System der ganzen Zahlen (und noch andere Zahlensysteme) bei Verf. ausgeschlossen sind. Dagegen behandelt Lenz die Exponentiation nicht, und der vorliegende Aufsatz hat den Vorzug grösserer Durchsichtigkeit und Übersichtlichkeit der Darstellung und weist auch originelle Sätze und Wege zu denselben auf. Es ist schade, dass Verf. nachdem er vom Aufsatz von Lenz Kenntnis genommen hat, nicht in seinem Aufsatz der somit gegebenen neuen Situation Rechnung getragen hat.

B. Germansky (Berlin)

SET THEORY

See also 10926.

10914:

Fraenkel, Abraham A. ★Abstract set theory. 2nd, completely revised edition. Studies in Logic and the

Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1961. viii + 295 pp. $8.00.

From the author's preface: "In this second edition [1st ed. (1953); MR **15**, 108] the arrangement in general lines has remained unchanged. However, essential changes of detail on almost every page have been made for several reasons. The text underwent a complete revision which renders the first sections more concise; many remarks of secondary importance and references to literature of minor significance were dropped; for some matters of principle the reader is now referred to the author and Y. Bar-Hillel's *Foundations of set theory* [North-Holland, Amsterdam, 1958; MR **21** #648]. Part of the space saved was used to add new material in the text and the exercises and to utilize pertinent publications which appeared in the decade 1950–1959. To a still higher degree the bibliography has been reduced. In spite of the new material, the size of the book has diminished from 479 to 295 pages. The procedure of the definition by transfinite induction has been fully formalized."

The author now uses the more conventional symbols for the union and intersection of sets, but introduces a notation for the set of all subsets of a set which is often employed for the complement of a set; he has retained his unconventional notation for cardinal numbers. There are many footnotes. The English is better than in the first edition, although not as good as a book of this caliber deserves, especially in view of the nature of the set of its intended readers. *F. Bagemihl* (Detroit, Mich.)

10915:

Matsuzaka, Kazuo. On the definition of the product of ordinal numbers. Sûgaku 8 (1956/57), 95–96. (Japanese)

Let μ be an ordinal number and $\{\xi_\alpha;\ \alpha<\mu\}$ be a well-ordered set of ordinal numbers ($\geqq 1$). The author gives a direct definition of $\prod_{\alpha<\mu}\xi_\alpha$ as follows: For every α, take a well-ordered set $M_\alpha=(a_\alpha,\ \cdots,\ x_\alpha,\ \cdots)$, whose ordinal number is ξ_α {originally α, corrected by the reviewer}. Then, $\prod_{\alpha<\mu}\xi_\alpha$ is the ordinal number of the well-ordered set, whose members are members $x=(\cdots,\ x_\alpha,\ \cdots)$ of the direct product $\prod_{\alpha<\mu}M_\alpha$ satisfying $x_\alpha=a_\alpha$ except for finite numbers of α's, and whose ordering is given anti-lexicographically. It is shown that this definition is equivalent to the ordinary recursive definition. Applicability of this definition is shown by a short proof of the existence and uniqueness of the ξ-adic expansion of ordinal numbers and a smart proof of Hessenberg's theorem $\aleph_\mu{}^2=\aleph_\mu$. *K. Ono* (Nagoya)

COMBINATORIAL ANALYSIS

10916:

Andress, W. R. Basic properties of pandiagonal magic squares. Amer. Math. Monthly **67** (1960), 143–152; correction, 658.

If a_{rs} is the number in row r and column s of any pandiagonal magic square of order $2n$, it has been conjectured by N. Chater and W. J. Chater [Math. Gaz. **29** (1945), 92–103; **33** (1949), 94–98; MR **11**, 229] that $a_{rs}+a_{r+n,s+n}=\text{const}$. It is shown here that this is true only for $n=2$. A detailed study of the squares of orders 4 and 5, with "magic number" equal to zero, produces general forms almost identical with those appearing in M. Kraitchik, *La mathématique des jeux* [Brussels, 1930]. Finally it is shown that the number of independent elements in a zero-sum pandiagonal magic square of order n is at most $(n-1)(n-3)$ if n is odd, and $(n-1)(n-3)+1$ if n is even. *J. Riordan* (New York)

10917:

Greenwood, Robert E. Linear graphs and matrices. Texas J. Sci. **12** (1960), 105–108.

Expository.

10918:

Čulík, Karel. Zu einer extremalen Aufgabe über die chromatischen Zahlen der endlichen Graphen. Časopis Pěst. Mat. **85** (1960), 14–17. (Czech. Russian and German summaries)

The author studies the change in the chromatic number of a finite unoriented graph when further edges are added to the graph. In particular, he proves that there exist k-chromatic graphs which become n-chromatic after the addition of $(n-k[n/k])[n/k]+k\binom{[n/k]}{2}$ edges.

H. Halberstam (London)

10919:

Read, Ronald C. A note on the number of functional digraphs. Math. Ann. **143** (1961), 109–110.

The author answers a question of the reviewer [Math. Ann. **138** (1959), 203–210; MR **22** #18] by reducing his formula for the number of functional digraphs expressed in terms of the cycle indexes of cyclic groups to the following, in which $T(x)$ is the generating function for rooted trees and $v(x)$ that for functional digraphs:

$$v(x)=\frac{x}{T(x)}\prod_{i=1}^{\infty}\{1-T(x^i)\}^{-1}.$$

F. Harary (Ann Arbor, Mich.)

10920:

Minty, G. J. Monotone networks. Proc. Roy. Soc. London. Ser. A **257** (1960), 194–212.

Author's summary: "Fundamental existence and uniqueness theorems for electrical networks of non-linear resistors are proved in an abstract form, as theorems of pure mathematics. The two groups from which the 'currents' and 'voltage drops' are drawn are permitted to be either the real numbers, or discrete subgroups of the reals. It is found that the uniqueness theory is derivable from extremum principles for certain convex functions associated with the networks, and that the existence theory is derivable from a single new theorem of graph theory.

"The abstract approach, besides revealing the logical structure of the subject more clearly than the 'concrete' approach, also (1) reveals the mathematical problem of solving a non-linear network to be identical with certain extremum problems arising in non-electrical applications, (2) contributes a numerical method, since the constructions for the discrete case are algorithmic, and (3) permits the

application of the theorems to problems of pure mathematics."

The single new theorem of graph theory is as follows. Let N be a network (directed graph) whose branches (directed lines) are partitioned into three sets A, B, and C, and let one branch b of the set B be distinguished. Then N contains either a cycle containing b but no line of C in which all lines of B are similarly directed, or a cocycle containing b but no line of A in which all lines of B are similarly directed. This theorem, in addition to its intrinsic interest in graph theory, leads to important applications in both linear and nonlinear programming and the steady-state solution of nonlinear electric networks (the multitude of definitions required for presenting these consequences does not permit their inclusion here).

F. Harary (Ann Arbor, Mich.)

10921:

Watanabe, Hitoshi. A method of tree expansion in network topology. IRE Trans. **CT-8** (1961), 4–10.

The sum of the tree admittance products of a graph is determined using a decomposition of the graph by independent cut sets. A method for finding an efficient decomposition is given. The procedure is suitable for use with a digital computer. *R. Cohn* (New Brunswick, N.J.)

10922:

Hakimi, S. L. On the realizability of a set of trees. IRE Trans. **CT-8** (1961), 11–17.

The author gives two simple necessary conditions for an incidence matrix to be the tree matrix of a graph and conjectures bounds for the number of trees in a graph which, if correct, furnish additional necessary conditions. From an incidence matrix which is a tree matrix he shows how the fundamental circuit and cut-set matrices may be obtained, and hence the graph itself. He shows that the number of separable parts of a graph is determined by the rank of the modified tree matrix.

R. Cohn (New Brunswick, N.J.)

10923:

Fulkerson, D. R. Zero-one matrices with zero trace. Pacific J. Math. **10** (1960), 831–836.

Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be given nonnegative integers. Conditions are found under which the constraints

$$\text{(1a)}\qquad \sum_{j=1}^{n} x_{ij} \leqq a_i \qquad (i = 1, \cdots, n),$$

$$\text{(1b)}\qquad \sum_{i=1}^{n} x_{ij} \geqq b_j \qquad (j = 1, \cdots, n),$$

$$\text{(1c)}\qquad x_{ij} = 0, \quad i = j, \qquad = 0 \text{ or } 1, \quad i \neq j,$$

have a solution x_{ij}. If $\sum a_i = \sum b_j$, then a solution is equivalent to a (0, 1)-matrix of order n having row sums a_i, column sums b_j, and zero trace. General feasibility conditions for the conditions (1) are deduced from the supply-demand theorem for network flows. These conditions simplify considerably if it is assumed that

$$\text{(2)}\qquad a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0, \quad b_1 \geqq b_2 \geqq \cdots \geqq b_n \geqq 0.$$

In fact, in this case, the conditions reduce to the n inequalities

$$\text{(3)}\qquad \sum_{i=1}^{k} b_i \leqq \sum_{i=1}^{k} a_i^{**} \qquad (k = 1, \cdots, n),$$

where a_j^{**} is the number of a_i such that $i<j$ and $a_i \geqq j-1$ plus the number of a_i such that $i>j$ and $a_i \geqq j$. A subsequent paper by the reviewer studies the latter problem for maximal and minimal trace, but by very different methods [Canad. J. Math. **12** (1960), 463–476; MR **22** #6723]. *H. J. Ryser* (Columbus, Ohio)

10924:

Erdös, P.; Rényi, A. On random graphs. I. Publ. Math. Debrecen **6** (1959), 290–297.

Let $\Gamma_{n,N}$ denote a random graph obtained by forming N links between n labelled vertices, all $\left(\binom{n}{2} \atop N\right)$ graphs being equiprobable. Let $P_k(n, N)$ denote the probability that the greatest connected part of $\Gamma_{n,N}$ contain exactly $n-k$ points. Then, denoting by N_c the integral part of $\frac{1}{2}n \log n + cn$ (c arbitrary, fixed), the authors prove that $P_k(n, N_c)$ tends, as $n \to \infty$, to $\gamma^k e^{-\gamma}/k!$ with $\gamma = e^{-2c}$. The same result obtains for $\pi_k(n, N_c)$, the probability that Γ_{n,N_c} consist of exactly $k+1$ disjoint connected components. These results extend much weaker previously known ones. The key to the proofs is given by the lemma: Let $P(A, n, N)$ denote the probability that all nonisolated points of $\Gamma_{n,N}$ (i.e., those with which at least one edge is associated) are connected, then $P(A, n, N_c) \to 1$ as $n \to \infty$. *A. Dvoretzky* (Jerusalem)

10925:

Erdös, P. Graph theory and probability. II. Canad. J. Math. **13** (1961), 346–352.

As in an earlier paper with the same title [same J. **11** (1959), 34–38; MR **21** #876], the author is concerned with the number $f(k, l)$, which is the least number such that every graph having $f(k, l)$ vertices contains either a complete graph of order k or a set of l independent vertices (a set of l vertices is called independent if no two are connected by an edge). The present paper is devoted to a proof, of probabilistic nature, of the relation $f(3, l) > cl^2 (\log l)^{-2}$, with c a positive constant.

J. Riordan (New York)

ORDER, LATTICES

10926:

Blair, R. L.; Tomber, M. L. The axiom of choice for finite sets. Proc. Amer. Math. Soc. **11** (1960), 222–226.

A certain class $\mathscr{F}$ of partially ordered sets (p.o. sets) is defined which are called p.o. sets with finitary covers. Among others, the authors consider these propositions: (I) If $P \in \mathscr{F}$ and every chain in P has a l.u.b., then P has a maximal element; (II) If $P \in \mathscr{F}$ and every chain in P has an upper bound, then P has a maximal element; (III) If $P \in \mathscr{F}$, then every chain in P can be extended

to a maximal chain. Both (II) and (III) are shown to be equivalent to the axiom of choice. On the other hand, (I) proves to be equivalent to the axiom of choice restricted to (arbitrary) families of non-empty finite sets. Only two examples of p.o. sets in $\mathscr{F}$ are given: (i) the set of all choice functions for sub-families of a family of non-empty finite sets partially ordered by the relation of being an extension; and (ii) the set $\mathscr{C} \cup \{P\}$ ordered by inclusion, where $\mathscr{C}$ is the collection of all chains of a given p.o. set P.

Dana Scott (Berkeley, Calif.)

10927:

Čulík, Karel. Über die Homomorphismen der teilweise geordneten Mengen und Verbände. Czechoslovak Math. J. **9** (84) (1959), 496–518. (Russian summary)

Eine Teilmenge P der (t, g)-Menge (teilweise geordnete Menge) M heisst eine eingelegte Teilmenge in M, wenn $z \in M-P$ impliziert xRz für jedem $x \in P$, wo R eine der drei Relationen $<$ oder $>$ oder $\|$ist $(x)\|z$ genau wenn x, z unvergleichbar sind).

Eine Abbildung ϕ der (t, g)-Menge M auf eine (t, g)-Menge N erfüllt sicher die Bedingung: $x, y \in M, x=y= \Rightarrow \phi(x)=\phi(y)$. Andere Bedingungen die sie erfüllen möchte sind:

(3) $x, y \in M, x<y \Rightarrow \phi(x)<\phi(y)$,

(4) $x, y \in M, x\|y \Rightarrow \phi(x)\|\phi(y)$,

(3′) $x, y \in M, x<y \Rightarrow \phi(x) \leq \phi(y)$,

(4′) $x, y \in M, x\|y \Rightarrow$ entweder $\phi(x)\|\phi(y)$ oder $\phi(x)=\phi(y)$.

(3) und (4) charakterisieren die Isomorphismen; (3), (4′) geben ein Graphenhomomorphismus. Ein A-, B-, oder C-Homomorphismus definiert man als eine Abbildung ϕ die die Bedingungen (3′), (4), bzw., (3), (4′), bzw. (3′), (4′) erfüllt.

Nennen wir eine (t, g)-Teilmenge N einer (t, g)-Menge M eine Kette [eine Gegenkette = 'antichain'] wenn $x\|y$ $[x<y]$ niemals erfüllt sei, dann haben wir die Sätze:

Notwendig und hinreichend dafür, dass eine Abbildung ϕ der (t, g)-Menge M auf eine (t, g)-Menge N ein A-, B-, oder C-Homomorphismus sei, ist die Bedingung dass die durch ϕ erzeugte Zerlegung $\bar{M}$ auf M eine Zerlegung in 'eingelegten Ketten', bzw. in 'eingelegten Gegenketten', bzw. in 'eingelegten (t, g)-Teilmengen', ist. $\bar{M}$ lässt sich jedesmal als eine (t, g)-Faktormenge geordnet ($P<Q$ in $\bar{M} \Rightarrow (p \in P, q \in Q \Rightarrow p<q)$).

M ist A- [B-, oder C-] einfach wann jedes A- [B-, C-] homomorphen Bild von M entweder eines isomorphen Bild oder eine einpunktige Menge ist. Die notwendige und hinreichende Bedingung dafür ist dass jede in M eingelegte Kette [Gegenkette, (t, g)-Teilmenge] nur ein einziges element enthält.

Jede (t, g)-Menge M hat eines (bis auf Isomorphie eindeutig bestimmtes) A- [B-, oder C-] homomorphen Bild, das A-[B-, C-] einfach ist. Dieses Bild AM zusammen mit die Ordnungstypen der Ketten die den Urbilder der einzigen Elementen von AM bei den A-Homomorphismus sind, charakterisiert M bis auf Isomorphie; ebenso kann man A charakterisieren bei BM und die Kardinalzahlen die Urbilder des einzigen Elemente.

Beispielen zeigen, dass eine A-einfache Menge nicht B-einfach sein möge, und umgekehrt. Die C-einfache Mengen haben keine nicht-triviale Zerlegungen in eingelegten (t, g)-Teilmengen (d.h. Archimedische Zerlegungen).

Endlich zeigt der Verfasser dass ein A-Homomorphismus das einen Verband auf einen Verband abbildet ein Verbandshomomorphismus sein muss. Und ein C-homomorphes Bild ein Verbandes ist wieder ein Verband.

Die verschiedene Möglichkeiten sind mit Beispielen illustriert.

V. S. Krishnan (Madras)

10928:

Raney, George N. Tight Galois connections and complete distributivity. Trans. Amer. Math. Soc. **97** (1960), 418–426.

This paper describes a class of Galois connections and its relation to complete distributivity in complete lattices. A method for constructing Galois connections between complete lattices is presented. These connections are called tight Galois connections, characterized by certain identities. All closure operations on complete lattices are obtainable from tight connections. If either of the complete lattices in a Galois connection is completely distributive, then the connection is tight. The identity mapping of a complete lattice to its dual is tight if and only if the lattice is completely distributive; this leads to a characterization of completely distributive complete lattices solely in terms of the partial order.

O. Ore (New Haven, Conn.)

10929:

Benado, Mihail. Bemerkungen zur Theorie der Vielverbände. IV. Über die Möbius'sche Funktion. Proc. Cambridge Philos. Soc. **56** (1960), 291–317.

[Previous parts appeared in Math. Nachr. **20** (1959), 1–16; MR **22** #19.] First are considered some properties of hierarchies (which the author defines as partially ordered sets in which all quotients are finite, differing from some other definitions) and of multilattices. If M is a semimultilattice, a special closure operation or its dual is definable. Properties of the Möbius function on a hierarchy are investigated, in particular conditions for the function to take the value zero. Also studied is the "Cartesian interpolation property" possessed by some multilattices and by all distributive multilattices. All complemented distributive multilattices are Boolean algebras.

Many of the definitions and results presented here in detail have previously been announced [e.g., C. R. Acad. Sci. Paris **246** (1958), 863–865, 2553–2555; **251** (1960), 622–623; MR **20** #6370, 6371; **22** #7962].

P. M. Whitman (Silver Spring, Md.)

10930:

Crawley, Peter. Decomposition theory for nonsemimodular lattices. Trans. Amer. Math. Soc. **99** (1961), 246–254.

Dilworth and Crawley [same Trans. **96** (1960), 1–22; MR **22** #9461] developed a theory of meet-decompositions in compactly-generated atomic lattices, mostly assuming semimodularity. The present paper extends the theory to nonsemimodular lattices. The elements of any compactly-generated atomic lattice have irredundant decompositions into completely irreducible elements, and this theorem implies the Axiom of Choice. A modified form of the semimodular law is found necessary and sufficient for the replacement property to hold in such a lattice.

P. M. Whitman (Silver Spring, Md.)

10931:

von Neumann, John. ★Continuous geometry. Foreword by Israel Halperin. Princeton Mathematical Series, No. 25. Princeton University Press, Princeton, N.J., 1960. xi + 299 pp. $7.50.

This book reproduces the lecture notes of the author [*Continuous geometry, I, II–III*, Inst. Advanced Study, Princeton, N.J., 1936, 1937]. Part I of this book discusses the axioms for continuous geometry and gives the construction of the numerical dimension function for the irreducible case. I. Kaplansky [Ann. of Math. (2) **61** (1955), 524–541; MR **19**, 524; particularly 537, 538] observed that almost all results of Chapter V hold in the complemented modular complete lattices without the continuity axioms of lattice operations. The main contents of Part II of this book are the properties of regular rings and the proof of the coordinatization theorem. The author showed that every complemented modular lattice of order $\geqq 4$ can be represented as the lattice of all principal right ideals of a regular ring. H. Löwig pointed out that there are two slips in the original note. One of them is in the proof of Lemma 11.6. In the present book Lemma 11.6 is changed slightly so that it is a reformation of Lemma 11.5 interchanging two indices. The second slip is in the very long proof of Theorem 13.1. In the present book this theorem is proved in a few words, using the following Lemma 13.2 (found by Halperin and von Neumann in 1937): Supposing $a \leqq b$ in a complemented modular lattice L, for any x in L there exists a relative complement u of a in b such that $x = (x \cup u) \cap (x \cup a)$. This is the only serious change in this book from the original notes. Part III of the present book is concerned with reducible continuous geometries. The author used central envelopes to obtain the fundamental comparability theorem. Using this theorem he began the construction of the dimension functions. At this point the present book breaks off as the original note. A completion of the author's dimension discussion which leads to a vector-valued dimension function and to a sub-direct decomposition of the lattice itself, has been given by T. Iwamura [Japan. J. Math. **19** (1944), 57–71; MR **8**, 35]. Cf. the reviewer's book *Kontinuierliche Geometrien* [Springer, Berlin, 1958; MR **19**, 833; Kapitel V].

In printing this book, the original notes are carefully read, and many parts are slightly altered. These changes, together with some comments, are listed at the back of the book. The lecture notes reproduced in the present book do not cover all of the author's work on continuous geometry and its related topics, but a survey of his other published and unpublished work on these topics is given in a foreword by I. Halperin. At the back of the present book, I. Halperin gives a few comments about the articles of other authors on continuous geometry and related topics. *F. Maeda* (Hiroshima)

10932:

Jónsson, Bjarni. Representations of complemented modular lattices. Trans. Amer. Math. Soc. **97** (1960), 64–94.

This paper presents a generalization of Theorem 14.1 in J. von Neumann's book *Continuous geometry* [#10931], the celebrated coordinatization theorem for complemented modular lattices.

A historical comment may be in order here. It is often said that von Neumann successfully generalized the following classical theorem: Suppose P is a projective geometry of dimension $n-1$ and either (i) $n \geqq 4$ or (ii) $n=3$ and Desargues' theorem holds in P. Then there exists a division ring D (not necessarily commutative) such that the points of P can be coordinatized by right-homogeneous coordinates $(x^1t, \cdots, x^nt)$, with t and all x^i in D, so that the lines, planes, etc., of P correspond to the solution-sets of systems of homogeneous equations of the form $c_1x^1 + \cdots + c_nx^n = 0$ with coefficients c_i all in D.

This "classical" theorem has long been accepted as part of mathematical folklore and reference is made to Hilbert and to Veblen and Young. But, in fact, the proof of Veblen and Young in Chapter VII of their *Projective geometry, Vol. I* [Ginn, Boston, 1916] assumes a condition which restricts P to the case where D is commutative. The proof by Hilbert in § 27 of his *Grundlagen der Geometrie* [8te Aufl., Teubner, Stuttgart, 1956; MR **18**, 227] covers (ii), that is, the Desarguesian plane, but not (i).

Von Neumann himself refers to the projective-geometry theorem as well known on page 63 of his *Continuous geometry* [op. cit.], but gives no references other than Veblen and Young. The footnote 30 to the paper by Birkhoff and von Neumann [Ann. of Math. (2) **37** (1936), 823–843, p. 834] refers to this situation but gives no precise reference to the literature.

A proof of an equivalent theorem is given by R. Baer on page 302 of his *Linear algebra and projective geometry* [Academic Press, New York, 1952; MR **14**, 675] but it is not clear that an acceptable proof was published before 1937 when von Neumann proved this projective-geometry theorem as a special case of his general theorem for complemented modular lattices.

Von Neumann's theorem assumes that the complemented modular lattice L has a basis $a = (a_1, \cdots, a_n)$ (this means that the a_i are independent and their union is the unit of L), such that the basis is homogeneous (this means that a_1 is perspective to a_i for all i) and $n \geqq 4$. An extension to the case $n=3$ plus a Desarguesian condition was first announced by the author [Bull. Amer. Math. Soc. **60** (1954), 24] but has not been published previously. Independently, a detailed proof for such an extension of von Neumann's theorem (i.e., to the lattice generalization of (ii)) was given by Fryer and Halperin [Acta Sci. Math. Szeged **17** (1956), 203–249; MR **20** #12], who used methods differing at some essential points from those of von Neumann.

In these papers the coordinatization theorem breaks into two parts: first (what appears to be the easier part) the ring S of coordinates is constructed (the division ring D has to be replaced by a more general ring, defined by von Neumann and called, by him, regular); secondly, a proof is given of the fact that the vectors $(x^1, \cdots, x^n)$ with all x^i in S can be used to coordinatize the elements of the given lattice L, or equivalently (as emphasized by von Neumann): L is isomorphic to the lattice of all principal right ideals of S_n, the ring of all $n \times n$ matrices over S.

In the paper under review, the author weakens the previous hypotheses on L by requiring of the basis only that $n \geqq 3$, that each a_i be perspective to some $a_i' \leqq a_1$, with $a_i' = a_1$ for $i = 2, 3$, and that a Desarguesian condition hold in L (the Desarguesian condition holds automatically if $a_i' = a_1$ for $i = 2, 3, 4$).

To prove this theorem the author gives (essentially) the following "extension lemma": Suppose A' is a complemented modular lattice with a basis $(a_0, a_1, a_2, a_3, \cdots, a_n)$, $n \geqq 3$, such that a_0 is perspective to each of a_1, a_2, a_3 (no restriction is placed on a_i for $i > 3$). Suppose A is the sublattice of A': $0 \leqq x \leqq a_1 + \cdots + a_n$. Suppose B is a complemented modular sublattice of A such that $a_1, \cdots, a_n$ are all in B and such that the mutual perspectivity of a_1, a_2, a_3 can be realized in B. Then there exists a complemented modular sublattice B' of A' such that: the common elements of B' and A are precisely B, B' contains a_0 and a_0 is perspective to a_1 in B' (in fact, if $a_{0,1}$ is any fixed element satisfying $a_{0,1} \oplus a_0 = a_{0,1} \oplus a_1 = a_0 \oplus a_1$, then B' can be taken as the smallest sublattice of A' which contains a_0, $a_{0,1}$ and includes B).

Now the lattice L of the original theorem is isomorphic to such a B inside such an A (itself inside such an A'). This is shown by the author by using Frink's theorem to imbed L as a sublattice of a direct union A of projective spaces, and then applying the coordinatization theorem for projective spaces to represent A as a lattice of subgroups of an abelian group. After that, A' is easy to construct using group operations. Thus the "extension lemma" shows that L can be imbedded in a bigger complemented modular lattice L' with basis $(a_0, a_1, \cdots, a_n)$, with a new element a_0 perspective to a_1.

By successive such imbeddings it is possible to imbed L in a complemented modular lattice L'' so that L'' has a homogeneous basis (d_1, d_2, d_3, d_4) and L coincides with the sublattice $0 \leqq x \leqq d_1$.

Then all that is needed is the first part of von Neumann's original proof, the construction of the regular ring S of coordinates for L''; in fact, L is isomorphic to the lattice of all principal right ideals of this S (the uniqueness of this regular ring is shown).

This imbedding procedure of the author offers an alternative to the second part of von Neumann's original theorem. But, in fact, von Neumann's theorem (or something like Baer's proof) is already used in assuming the coordinatization theorem for projective spaces in order to apply the extension lemma to the original L. (Baer's proof itself uses an extension lemma for projective spaces; it would be reasonable to hope for a refinement of the author's methods which would prove his extension lemma without recourse to Frink and Baer, thus giving a self-contained proof of the author's generalization of von Neumann's theorem.)

The author also shows that his theorem coordinatizes every simple complemented modular lattice (excluding the non-Desarguesian cases). Then he varies one of von Neumann's constructions to display such a simple complemented modular lattice to which the previous coordinatization theorems do not apply, at least not directly.

The author's extension lemma is proved as follows: Let $\bar{a}_i$ denote $a_0 + a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n$. For $i = 2$, 3 choose $a_{1,i}$ in B so that $a_1 \oplus a_{1,i} = a_i \oplus a_{1,i} = a_1 \oplus a_i$. Choose $a_{0,1}$ in A' so that $a_0 \oplus a_{0,1} = a_1 \oplus a_{0,1} = a_0 \oplus a_1$, and for $i = 2, 3$ define $a_{0,i} = (a_{0,1} \oplus a_{1,i})(a_0 \oplus a_i)$.

A is mapped onto $0 \leqq x \leqq \bar{a}_i$ by the perspectivity mapping with axis $a_{0,i}$. This perspectivity mapping can be extended to a lattice automorphism α^i of A'. (The author uses the group representation of L to construct α^i, but it can be done abstractly with the method of Amemiya, J. Math. Soc. Japan 9 (1957), 263–279 [MR 19, 1154].)

Now let B_i denote the image of B under α^i $(i = 1, 2, 3)$, and let $B_{i,j}$ $(i, j = 1, 2, 3, i \neq j)$ denote the set of all elements in A' of the form

$$c = x + (a_i + y)(a_j + z)$$

with x in B, y in B_i, z in B_j, $y\bar{a}_0 = z\bar{a}_0 = 0$. It is easy to see that: each $B_{i,j}$ is complemented, a_0 is in $B_{i,j}$, and the elements common to A and $B_{i,j}$ are precisely those of B. To complete the proof of the extension lemma, it is only necessary to show that some $B_{i,j}$ is a sublattice of A', that is, (α) if c, d are in $B_{i,j}$, then so is $c + d$, (β) if c, d are in $B_{i,j}$, then cd is the union of a finite number of elements in $B_{i,j}$ (then use (α)).

{The reviewer notes that (α) can be easily reduced to proving (α'): if c, d are in $B_{i,j}$ with $cd = c\bar{a}_0 = d\bar{a}_0 = 0$, then $(c+d)\bar{a}_0$ is in B.} The proofs of (α), (β) (or of (α'), (β)) are the difficult steps in the present paper. They are obtained by first showing that all $B_{i,j}$ are identical and that without changing $B' = B_{i,j}$, the following can be done: (I) leaving a_j fixed for $j \neq i$, a_i can be moved inside $\bar{a}_0$ but outside $\bar{a}_0\bar{a}_i$; (II) leaving a_j fixed for $j \neq 0$, a_0 can be moved outside $a_1 + \cdots + a_n$.

{The reviewer remarks that the paper of Fryer and Halperin [op. cit.], suitably modified in minor details, proves the author's coordinatization theorem.}

I. Halperin (Kingston, Ont.)

GENERAL MATHEMATICAL SYSTEMS

See also 11060.

10933:

Fujiwara, Tsuyoshi. On mappings between algebraic systems. II. Osaka Math. J. 12 (1960), 253–268.

The paper continues the work of the author begun in same J. 11 (1959), 153–172 [MR 22 #1535], whose language is used. This paper is referred to as (I). An equivalence relation (B_W-conjugacy) among basic mapping formulas for systems with compositions W is introduced, relative to a system of composition identities B_W; it corresponds to the relation of "inner" isomorphism between the algebraic structures determined on $\mathfrak{B}^m$ relative to the different mapping formulas as in (I), where $\mathfrak{B}$ is an arbitrary B_W-system. The notions of homomorphism-type and derivation-type for mapping formulas are defined using this relation of conjugacy; they are investigated in some detail where the compositions and identities are those of commutative linear algebras over a given field K of characteristic zero.

G. B. Seligman (New Haven, Conn.)

10934:

Hasse, Maria. Einige Bemerkungen über Graphen, Kategorien und Gruppoide. Math. Nachr. 22 (1960), 255–270.

The notion of a graph is redefined abstractly. The notions of free category and groupoid generated by a graph—the latter is, more or less, the fundamental groupoid—are introduced. Categories and groupoids are quotients of associated free ones. Criteria for freeness are given.

A. Heller (Urbana, Ill.)

CLASSICAL ALGEBRA

10935:

Gaede, Karl-Walter. Zur Verteilung der Wurzeln zufälliger algebraischer Gleichungen. Math. Nachr. **21** (1960), 81–107.

Let $X_\nu = U_\nu + iV_\nu$ $(\nu = 1, \cdots, n)$ be the roots of $x^n + (S_1 + iT_1)x^{n-1} + \cdots + (S_n + iT_n) = 0$ $(U_\nu, V_\nu, S_\nu, T_\nu$ real) arranged according to decreasing real part and (in case of equal real parts) according to increasing imaginary part. If the coefficients have a probability distribution with joint density $h(s_1, t_1, \cdots, s_n, t_n)$ then the roots are distributed according to the joint density

$$\varphi(u_1, v_1, \cdots, u_n, v_n) =$$
$$h(s_1, t_1, \cdots, s_n, t_n) \prod_{\nu=1}^{n-1} \prod_{\mu=\nu+1}^{n} [(u_\nu - u_\mu)^2 + (v_\nu - v_\mu)^2],$$

where $s_\nu + it_\nu$ is the νth elementary symmetric function of $u_1 + iv_1, \cdots, u_n + iv_n$. Similar results are proved also for equations with real coefficients possessing a joint density. The cubic equation is studied in detail and special cases are illustrated. Earlier work is reviewed.

A. Dvoretzky (Jerusalem)

10936:

Lewis, F. A. Circulants and their groups. Amer. Math. Monthly **67** (1960), 258–260.

The author determines the group of all permutations of $x_0, x_1, \cdots, x_{n-1}$ which leave the circulant $\det(x_{i-j})$ unaltered. The order of the group is $n\phi(n)$ if n is odd and $\frac{1}{2}n\phi(n)$ if n is even.

H. K. Farahat (Sheffield)

10937:

Lehti, Raimo. Eine neue Lösungsmethode für die algebraische Gleichung vierten Grades. Soc. Sci. Fenn. Comment. Phys.-Math. **25** (1960), no. 4, 8 pp.

Let $p(x)$ and $q(x)$ be two polynomials over a field K with m roots λ_i and n roots μ_j in an algebraic extension of K. A polynomial over K, which has exactly mn roots $\lambda_i + \mu_j$, is defined as algebraic sum of $p(x)$ and $q(x)$. Similarly a polynomial over K, which has exactly mn roots $\lambda_i\mu_j$, is defined as algebraic product of $p(x)$ and $q(x)$. A polynomial $r(x)$ over K, which is an algebraic sum or product of two polynomials over K with degrees less than that of $r(x)$, is called sum-reducible or product-reducible over K respectively. The author determines conditions in order that a given polynomial of fourth degree be sum-reducible or product-reducible. He further shows that any polynomial of fourth degree can be transformed into a product-reducible one by replacing the ground field K by a cubic extension of K. This yields a variant of the classical method for solving algebraic equations of fourth degree. It should be noted that the author pays no attention to the characteristic p of K in the course of his proof and that part of his results is not valid when $p = 2$.

E. Inaba (Tokyo)

THEORY OF NUMBERS

See also 10983.

10938:

Бухштаб, А. А. [Buhštab, A. A.]. ★Теория чисел [Number theory]. Gosudarstv. Uč.-Ped. Izdat., Moscow, 1960. 375 pp. 7.60 r.

This is a textbook for a first course in number theory in universities and pedagogical institutes. It consists of 36 chapters, covering the usual material on congruences, linear and quadratic diophantine equations, and continued fractions, as well as (for example) the transcendence of e and π, Liouville's theorem, representations by binary quadratic forms, mean values of number-theoretic functions, and theorems of Čebyšev and Meissel on distribution of primes. Most chapters are followed by illuminating historical comments. The style is leisurely, and there are many numerical examples. There are no exercises.

W. J. LeVeque (Ann Arbor, Mich.)

10939:

Pisot, Charles. ★Les nombres entiers—leurs problèmes et leurs mystères. Université de Paris. Les Conférences du Palais de la Découverte, Série A, No. 261. Edition du Palais de la Découverte, Paris, 1960. 18 pp. 1.50 NF.

An expository article for the non-specialist, concerning the methods and problems of number theory. The author touches on diophantine equations, the Goldbach conjecture, Roth's theorem, transcendental numbers, distribution of primes, and p-adic analysis.

10940:

Sierpiński, W. On some unsolved problems of arithmetics. Scripta Math. **25** (1960), 125–136.

In this printed version of a lecture there are listed many unsolved problems on diophantine equations, distribution of primes and divisibility properties of integers. Most of the problems have been raised before and are known to be extremely difficult, but a new anthology calling attention to these prominent gaps in our knowledge is always a valuable addition to mathematical literature.

H. Halberstam (London)

10941:

Shanks, Daniel. On numbers of the form n^4+1. Math. Comput. **15** (1961), 186–189.

Let $Q(x)$ denote the number of primes of the form n^4+1 not exceeding x^4+1. This function is tabulated, from data given by A. Gloden, for $x = 100(100)1000$ and compared with the conjectured approximate formula

$$Q(x) \doteq .66974 \int_2^x dt/\log t$$

where the constant is given by

$$\frac{1}{4} \prod \left\{ 1 - \frac{\chi(-1) + \chi(2) + \chi(-2)}{p-1} \right\}$$

Here $\chi(h)$ is the Legendre symbol (h/p) and the product is over odd primes. This slowly convergent product is evaluated by means of Dirichlet L-series and zeta-functions. The problem is compared with the corresponding one for the function n^2+1.

D. H. Lehmer (Berkeley, Calif.)

10942:

Miller, J. C. P. (Editor). ★Representations of primes by quadratic forms: displaying solutions of the Diophantine

equation $kp=a^2+Db^2$. Part I: $D=5$, 6, 10, and 13. Royal Society Mathematical Tables, Vol. 5. Prepared by Hansraj Gupta, M. S. Cheema, A. Mehta and O. P. Gupta. Cambridge University Press, New York, 1960. xxiv+135 pp. $8.50.

The forms considered in this volume are: a^2+Db^2 ($D=5$, 6, 10, 13). The primes are those less than 10^5. Primes p of the forms:

$$\begin{aligned} p &= 20m+1, 9 && \text{if } D = 5,\\ p &= 24m+1, 7 && \text{if } D = 6,\\ p &= 40m+1, 9, 11, 19 && \text{if } D = 10,\\ p &= 52m+1, 9, 17, 25, 29, 49 && \text{if } D = 13 \end{aligned}$$

can be uniquely represented by a^2+Db^2 and so there exists positive integers k, n such that $kp=n^2+D$. The tables give a, b, k, n. The doubles of primes p of the forms

$$\begin{aligned} p &= 20m+3, 7 && \text{if } D = 5,\\ p &= 24m+5, 11 && \text{if } D = 6,\\ p &= 40m+7, 13, 23, 37 && \text{if } D = 10,\\ p &= 52m+7, 11, 15, 19, 31, 47 && \text{if } D = 13 \end{aligned}$$

can be represented by a^2+Db^2 and in these cases the tables give a, b, k, n in $2p=a^2+Db^2$ and $2kp=n^2+D$. These tables greatly extend the data about these forms in Cunningham, *Quadratic partitions*, F. Hodgson, London, 1904.

An elaborate introduction of 22 pages gives the underlying theory of quadratic fields and their ideals. There is no mention of applications of the table to such things as cyclotomy or power residues.

The tables were prepared by hand and then punched on cards for verifying and printing. The tables are beautifully printed. *D. H. Lehmer* (Berkeley, Calif.)

10943:

Tull, J. P. Average order of arithmetic functions. Illinois J. Math. **5** (1961), 175–181.

Making use of a theorem of a previous paper [Pacific J. Math. **9** (1959), 609–615; MR **21** #6362] the author obtains estimates for the sums $\sum_{n<x}\mu_k(n)$, $\sum_{n<x}\mu_k(n)/n$, $\sum_{n<x}2^{\nu(n)}$, $\sum_{n<x}2^{\nu(n)}/n$, $\sum_{n<x}d(n^2)$, $\sum_{n<x}d(n^2)/n$, $\sum_{n<x}d(n)^2$, $\sum_{n<x}d(n)^2/n$, $\sum_{n<x}d_k(n)/n$, where μ_k is the characteristic function of the kth-power-free integers, $\nu(n)$ is the number of distinct prime factors of n, $d_k(n)$ is the number of solutions of $x_1x_2\cdots x_k=n$ and $d(n)=d_2(n)$.

L. Carlitz (Durham, N.C.)

10944:

Mann, Henry B. On modular computation. Math. Comput. **15** (1961), 190–192.

An iterative process using matrix multiplication techniques for finding the least non-negative residue x mod $M=m_1m_2\cdots m_r$ if $x\equiv x_i \pmod{m_i}$, $0\le x_i<m_i$, $(m_i, m_k)=1$ for $i\ne k$.

O. Taussky-Todd (Pasadena, Calif.)

10945:

Wall, D. D. Fibonacci series modulo m. Amer. Math. Monthly **67** (1960), 525–532.

The author proves that the residue modulo m of the Fibonacci series defined by $f_0=a$, $f_1=b$, $f_{n+1}=f_n+f_{n-1}$ is completely periodic and proves a number of results about the length of the period. For example, letting $k(p)$ denote the period of the residues (mod p) of the series obtained when $a=0$, $b=1$, he proves that if $k(p^2)\ne k(p)$ then $k(p^e)=p^{e-1}k(p)$. The author mentions that no value of p is known for which $k(p^2)=k(p)$ nor has it been proved that any such value exists.

W. H. Simons (Vancouver, B.C.)

10946:

Ząbek, Świątomir. Sur la périodicité modulo 10^μ des suites de nombres $S_p(n)$. Ann. Univ. Mariae Curie-Skłodowska Sect. A **13** (1959), 145–155. (Polish and Russian summaries)

In an earlier paper [same Ann. Sect. A. **10** (1956), 37–47 (1958); MR **20** #1653] the author found for a given positive integer m the length l of the shortest period modulo m of the sequence $\binom{n}{k}$, k a fixed positive integer, $n=1, 2, 3, \cdots$. The present paper examines the same question for the sequences $\rho_p(n)=\sum_{i=1}^n i^p$, p a fixed positive integer or 0, $n=1, 2, \cdots$, which are also periodic for any modulus m. The author derives for $m=10^\mu$, $\mu=1, 2, \cdots$, the length l_p of the shortest period modulo m for the cases $p=0, 1, \cdots, 9$. Using the Bernoulli expressions for $\rho_p(n)$, each p is treated individually by congruence considerations which become complicated with increasing p. The length l_p of the shortest period modulo 10^μ of $\rho_p(n)$, $n=1, 2, \cdots$, is given as follows: $l_0=10^\mu$; $l_1=l_2=l_6=2\cdot 10^\mu$; $l_4=l_8=10^{\mu+1}$; $l_3=l_7=l_9=10^\mu$ for $\mu\ge 2$, but $=20$ for $\mu=1$; $l_5=2\cdot 10^{\mu-1}$ for $\mu\ge 2$, but $=20$ for $\mu=1$. *A. J. Kempner* (Boulder, Colo.)

10947:

Rubel, L. A. An Abelian theorem for number-theoretic sums. Acta Arith. **6** (1960), 175–177.

The author considers pairs of functions $f(n)$, $f^*(n)$ connected by the equivalent relations

$$f(n) = \sum_{d|n} f^*(d), \quad f^*(n) = \sum_{d|n} f(d)\mu\left(\frac{n}{d}\right).$$

There are known Abelian and Tauberian theorems connecting convergence or limitability of the sequence $\{f(n)\}$ with convergence or summability of the series $\sum_1^\infty f^*(n)/n$ (with equality of limit and sum) [A. Wintner, *Eratosthenian averages*, Baltimore, Md., 1943; MR **7**, 366; §§ 8–12, §§ 18–24; G. H. Hardy, *Divergent series*, Clarendon, Oxford, 1949; MR **11**, 25; Appendix IV]. To these the author adds the Abelian theorem: If $f(n)\to L$ (finite) as $n\to\infty$, then $\sum_1^\infty f^*(n)/n$ converges to L. The author's argument is based on the claim that the conditions for regularity of the appropriate matrix transformation are equivalent to the known results (a) $N(k)\to 0$ as $k\to\infty$, (b) $\sum_{k=1}^\infty |N(k)|/k<\infty$, of prime number theory, where $N(x)=\sum_{k\le x}\mu(k)/k$. In a later correction, however, he notes that (as pointed out by P. T. Bateman) the argument is inadequate at the point where (b) is used. To repair the omission he applies the known theorem that $|N(k)|\log^2 k<H$, where H is a constant. This is stronger than (a) and (b), and is more than sufficient for regularity of the transformation.

A. E. Ingham (Cambridge, England)

10948:

Chidambara Swamy, J.; Venkateswara Rao, N. A further extension of a result of Mordell's. Math. Ann. **142** (1960/61), 244–253.

Erdös conjectured that for given $c \geqq 1$, the quotient $(2x)!/(x!(x+c)!)$ is integral for infinitely many integers x. This result has been generalized by Mordell [J. London Math. Soc. **34** (1959), 134–138; MR **21** #17], Wright [ibid. **33** (1958), 476–478; MR **21** #18] and McAndrew [Proc. Cambridge Philos. Soc. **55** (1959), 210–212; MR **21** #3372]. In the present paper the following are proved. Theorem I: Put $\psi(x)=(nx-r)!x/\prod_{i=1}^{m}(a_ix+c_i)$, where $r \geqq 1$, $1<m \leqq n$, $0<a_1<n$, $\sum a_i=n$, $c_i \geqq 0$. Then if $n/(a_1, n)>r$ and either $a_1/(a_1, n)>c_1$ or $n/(a_1, n)>\sum c_i$, it follows that $\psi(x)$ is integral for infinitely many integral x. Theorem II: (a) $(2x-1)!x/(x+1)!x!$ is non-integral for infinitely many x, (b) $(nx)!/(x!)^{n-1}(x+n+1)!$ is non-integral for infinitely many x, (c) $(nx)!/(x!)^{n-1}(x+n-1)!$ is integral for all sufficiently large x. (d) If $n+1$ is a prime or $n=3$, then $(nx)!/(n!)^{n-1}(x+n)!$ is non-integral for infinitely many x. (e) If $n+1$ is not a prime and $n \neq 3$, then $(nx)!/(n!)^{n-1}(x+n)!$ is integral for all sufficiently large x.

L. Carlitz (Durham, N.C.)

10949:

Karst, E. Faktorenzerlegung Mersennescher Zahlen mittels programmgesteuerter Rechengeräte. Numer. Math. **3** (1961), 79–86.

The author lists some recent discoveries, mostly by himself, of "small" prime factors of 2^p-1, where p is prime. The author's results are mostly in the distant range $10^5<p<10^8$ with factors less than 2^{32} and were obtained on the IBM 650 machine.

D. H. Lehmer (Berkeley, Calif.)

10950:

Apéry, Roger. Sur une équation diophantienne. C. R. Acad. Sci. Paris **251** (1960), 1451–1452.

The author considers the diophantine equation (1) $p^n=x^2+A$ $(n \geqq 0)$, where A is a positive integer and p is an odd prime not dividing A (both p and A are given in advance). He proves that (1) has at most 2 solutions. (The case $p=2$, $A=7$ was considered by Skolem, Chowla and Lewis in Proc. Amer. Math. Soc. **10** (1959), 663–669 [MR **22** #25]. In this case (1) has 5 solutions, where x is restricted to be positive.) *S. Chowla* (Boulder, Colo.)

10951:

Apéry, Roger. Sur une équation diophantienne. C. R. Acad. Sci. Paris **251** (1960), 1263–1264.

The diophantine equation (1) $2^{n+2}=x^2+A$ $(n \geqq 0)$, where A is a given positive integer congruent to 7 (mod 8), has at most 2 solutions in integers provided $A \neq 7$. (When $A=7$, it was proved that (1) has 5 solutions. See #10950.)

S. Chowla (Boulder, Colo.)

10952:

Lewis, D. J.; Mahler, K. On the representation of integers by binary forms. Acta Arith. **6** (1960/61), 333–363.

The authors improve earlier results of Davenport and Roth [Mathematika **2** (1955), 160–167; MR **17**, 1060] and Mahler [Math. Ann. **107** (1933), 691–730; **108** (1933), 37–55] concerning the number of solutions of the diophantine equation $F(x, y)=m$, where $F(x, y)$ is a form of degree n at least 3, with integral coefficients numerically smaller than a and with nonzero discriminant, and m is an integer different from 0. Specifically, it is shown that there are not more than $c_1(an)^{c_2\sqrt{n}}+(c_3n)^{t+1}$ pairs x, y with $x \neq 0$, $y>0$, and $(x, y)=1$, for which $F(x, y) \neq 0$ has at most t given prime factors $p_1, \cdots, p_t$. Here c_1, c_2, c_3 are positive absolute constants (of the order of magnitude of 100). It follows that if $|m|>m_0(a, n)$, then the number of primitive solutions of $F(x, y)=m$ is not greater than $(c_3n)^{t+1}$; this bound is independent of the coefficients of the form. The proof depends on the Mahler-Parry p-adic Thue-Siegel theorem.

As an application, an upper bound is obtained for the number of solutions of an equation of the form

$$p_{11}^{x_{11}}\cdots p_{1r}^{x_{1r}}+p_{21}^{x_{21}}\cdots p_{2s}^{x_{2s}} = p_{31}^{x_{31}}\cdots p_{3t}^{x_{3t}},$$

where the p_{ij} are fixed distinct primes.

W. J. LeVeque (Ann Arbor, Mich.)

10953:

Lubelski, S. Unpublished results on number theory. I. Quadratic forms in a Euclidean ring. Edited by C. Schogt. Acta Arith. **6** (1960/61), 217–224.

This is the first of a sequence of notes giving original results contained in an unpublished manuscript of the Polish mathematician S. Lubelski who met his death in World War II.

Let $\mathfrak{R}$ be the ring of rational integers or the ring of integers in one of the complex quadratic fields with a euclidean algorithm. It is shown that the number of classes of quadratic forms with coefficients in $\mathfrak{R}$ and with a given non-zero determinant and with a given number of variables is finite. {The reviewer notes that the corresponding result when $\mathfrak{R}$ is the ring of integers of any algebraic number field is Hilfssatz 40 of C. L. Siegel, Ann. of Math. (2) **38** (1937), 212–291.} The proof here gives an explicit upper bound for the least non-zero number represented by the form.

J. W. S. Cassels (Cambridge, England)

10954:

Sah, Chih-Han. Quadratic forms over fields of characteristic 2. Amer. J. Math. **82** (1960), 812–830.

This paper gives a solution to the integral equivalence problem for quadratic forms in any finite number of variables over a local field of characteristic 2. Let Ω denote a finite field of characteristic 2, and k the formal power series field over Ω with uniformizer π; $\mathfrak{o}$ is the ring of integral power series. By a quadratic lattice, M, the author means a free module of finite type over $\mathfrak{o}$ together with a map $Q: M \to k$ such that $Q(ax)=a^2Q(x)$ for $a \in k$ and $x \in M$, and such that $\langle x, y\rangle=Q(x+y)+Q(x)+Q(y)$ is bilinear. A quadratic lattice M is called an i-modular lattice if, when $\{x\} \subseteq M$ is a pure subset, $\langle x, M\rangle=\pi^i\mathfrak{o}$. Also $s(M)=\{\langle x, y\rangle \mid x, y \in M\}$ is called the scale of M and $q(M)=\{Q(x)+s(M) \mid x \in M\}$ is called the norm group of M.

One of the author's principal results is the following theorem. Let M_1 and M_2 be i-modular lattices. Then, $M_1 \cong M_2$ if and only if $kM_1 \cong kM_2$ and $q(M_1)=q(M_2)$.

Conditions also are established for equivalence of lattices in general but they are too involved to give here.

B. W. Jones (Boulder, Colo.)

10955:

Lehman, R. Sherman. On Liouville's function. Math. Comp. **14** (1960), 311–320.

Liouville's function $\lambda(n)$ is defined by $\lambda(n)=(-1)^r$ where r is the number of prime factors of n, multiple factors being counted according to their multiplicity. Pólya conjectured that the function $L(x)=\sum_{n\leq x}\lambda(n)$ was negative or zero for all $x>2$, but this conjecture was proved false by the reviewer [Mathematika **5** (1958), 141–145; MR **21** #3391]. In this paper the author confirms the reviewer's calculations of the sum

$$A_T^*(u)=\sum_{|\gamma_n|<T}\alpha_n\left(1-\frac{|\gamma_n|}{T}\right)e^{i\gamma_n u},$$

where γ_n runs through the imaginary parts of the zeros ρ_n of the Riemann zeta function and where $\alpha_0=1/\zeta(\frac{1}{2})$, $\alpha_n=\zeta(2\rho_n)/\rho_n\zeta'(\rho_n)$. For $T=1000$, $A_T^*(u)$ is found to be positive for $u=831.847$, taking the value 0.0050. This confirms the reviewer's result. However $A_T^*(u)$ is also found to be positive for $u=814.492$ where it takes the much larger value 0.0782. These results suggest, but do not prove, that $L(x)$ is positive for these values of $u=\log x$.

It was found that $A_T(u)=\sum_{|\gamma_n|<T}\alpha_n e^{i\gamma_n u}$ nearly became positive for $T=1000$ and $u=20.62$. This suggested that $L(x)$ might become positive for x near 9.05×10^8. By direct calculation of $L(x)$ it was found that $L(9061\ 80359)=+1$ and that $L(x)$ is positive over a considerable number of ranges of x in the intervals 9061 70000 to 9062 00000 and 9064 70000 to 9065 00000. Some of the 167 values of x found in the range for which $L(x)=0$ are listed.

C. B. Haselgrove (Manchester)

10956:

Briggs, W. E.; Buschman, R. G. The power series coefficients of functions defined by Dirichlet series. Illinois J. Math. **5** (1961), 43–44.

Let the Dirichlet series $f(s)=\sum_{n=1}^{\infty}h(n)n^{-s}$ have abscissa of convergence $\operatorname{Re} s=a$ and a simple pole at the point $s=a$. Then it possesses a Laurent expansion about $s=a$ of the form

$$f(s)=\frac{C}{s-a}+\sum_{r=0}^{\infty}\frac{(-1)^rC_r}{r!}(s-a)^r.$$

The purpose of the present note is to exhibit the coefficients C_r in terms of summatory functions involving $h(n)$. It is shown, in fact, that

$$C_r=\lim_{x\to\infty}\left\{\sum_{n\leq x}n^{-a}h(n)(\log n)^r-\frac{C}{r+1}(\log x)^{r+1}\right\}.$$

The special case of this result corresponding to $h(n)=1$ had been obtained previously by W. E. Briggs and S. Chowla [Amer. Math. Monthly **62** (1955), 323–325; MR **16**, 909].

L. Mirsky (Sheffield)

10957:

Buschman, R. G. Some infinite series for $\zeta(n+1)$. Amer. Math. Monthly **67** (1960), 260–263.

The author obtains some transformations of infinite series. Example:

$$\zeta(n+1)=\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}\frac{k^{n-1}+m^{n-1}}{(mk)^n}-\zeta(n)\right),$$

where $n(\geq 2)$ is a positive integer, and $\zeta(s)$ is Riemann's zeta function.

S. Chowla (Boulder, Colo.)

10958:

Staś, W. Über die Dichte der Nullstellen der Dirichletschen L-Funktionen. Acta Arith. **6** (1960/61), 313–323.

Denote by $N(\vartheta, T)$ the number of zeros of all $\phi(k)$ L-functions (mod k) where $k\geq 1$ in the region: $\sigma\geq\vartheta$, $\frac{1}{2}\leq\vartheta<1$, $0<t\leq T$ (multiplicity taken into account). Generalizing a result of P. Turán in his well-known book *Eine neue Methode in der Analysis und deren Anwendungen* [Akadémiai Kiadó, Budapest, 1953; MR **15**, 688], the author proves the following theorem: There exist positive constants b, c_1, c_2 such that

$$(1)\qquad N(\vartheta, T)<c_2k^6T^{2(1-\vartheta)+(1-\vartheta)^{1.09}}\log^8(kT)$$

holds for

$$(2)\qquad 1-\min(b, k^{-40})\leq\vartheta\leq 1$$

and

$$(3)\qquad T>\max(c_1, \exp\exp k^3).$$

The proof uses the interesting Satz X of Turán's book as well as results of Tatuzawa [Proc. Japan Acad. **26** (1950), no. 9, 1–13; MR **14**, 249] and Hua [Quart. J. Math. Oxford Ser. (2) **20** (1949), 48–61; MR **10**, 597].

S. Chowla (Boulder, Colo.)

10959:

Brewer, B. W. On certain character sums. Trans. Amer. Math. Soc. **99** (1961), 241–245.

Let p denote an odd prime and $\chi(f(x))$ the quadratic character of $f(x)$ with respect to p. The sum $\phi_k(Q)=\sum_{x=0}^{p-1}\chi(x(x^k+Q))$ has been studied by Jacobsthal [J. Reine Angew. Math. **132** (1907), 238–245]. Jacobsthal showed that if p is of the form a^2+b^2 (b even) and Q is a quadratic nonresidue of p, then $\phi_2(Q)=\pm 2b$. In the first half of this paper the author determines the sign of $\phi_2(-3)$ if $p=12k+5$. Specifically, he shows that $\phi_2(-3)=2b$, where $a\equiv 1 \pmod 4$ and $b\equiv a \pmod 3$. Moreover, he derives the congruence $\binom{6k+2}{k}\equiv 2b \pmod p$.

It is well known that p can be expressed in the form c^2+2d^2 if and only if $p=8k+1$ or $8k+3$. Using cyclotomy the author proves in the second half of the paper that

$$\sum_{x=0}^{p-1}\chi((x+2)(x^2-2))=0 \quad\text{if}\quad p\neq c^2+2d^2,$$
$$=2c \quad\text{if}\quad p=c^2+2d^2,$$

where $c\equiv(-1)^{k+1} \pmod 4$. He also indicates similar results for primes of the form u^2+5v^2.

A. L. Whiteman (Los Angeles, Calif.)

10960:

Shanks, Daniel. On the conjecture of Hardy & Littlewood concerning the number of primes of the form n^2+a. Math. Comp. **14** (1960), 320–332.

A famous and as yet unproved conjecture of G. H. Hardy and J. E. Littlewood [Acta Math. **44** (1923), 1–70] states that, if the integer a is not equal to a negative square, then the number $P_a(N)$ of primes of the form n^2+a with $1\leq n\leq N$ satisfies, for $N\to\infty$, the asymptotic relation

$$(*)\qquad P_a(N)\sim\frac{1}{2}\prod_{p\nmid a}\left\{1-\left(\frac{-a}{p}\right)\frac{1}{p-1}\right\}\int_2^N\frac{du}{\log u}.$$

It was verified by A. E. Western [Proc. Cambridge Philos. Soc. **21** (1922/23), 108–109] that, for $a=1$ and $N \leq 15{,}000$, the agreement between $P_a(N)$ and the right-hand side of (*) is good. In the present paper the discussion of the numerical evidence is taken a stage further. First of all the infinite product in (*) is transformed into a more rapidly converging product; this step facilitates its computation. Next, by considering the range $N \leq 180{,}000$, the author shows that (*) is almost certainly valid for $a=1, \pm 2, \pm 3, 4$. Finally, he presents a heuristic sieve method which suggests the validity of (*) for all values of a.

L. Mirsky (Sheffield)

10961:

Good, I. J. An asymptotic formula for the differences of the powers at zero. Ann. Math. Statist. **32** (1961), 249–256.

Though the title places emphasis on the differences of the powers at zero and hence on Stirling numbers of the second kind, a similar formula is also given for Stirling numbers of the first kind. Both formulas have been anticipated by L. Moser and M. Wyman [Duke Math. J. **25** (1957), 29–43; J. London Math. Soc. **33** (1958), 133–146; MR **19**, 1039; **20** #4664] (and possibly also, for the second kind, G. Arfwedson [Skand. Aktuarietidskr. **34** (1951), 121–132; MR **13**, 956]). However, the author has given two extensive tables of numerical values for roots of certain transcendental equations, essential in the use of the formulas. {Reviewer's note: For numbers of the second kind, the transcendental equation, the author's equation (4), is equivalent to $ue^{-u}=ae^{-a}$, with $a=1+\kappa$, $u=a-\rho$, and the root in question is such that $u \leq 1 \leq a$; for the numbers of the first kind, the corresponding result, equation (16), is equivalent to $ue^{-u}=a^{-1}e^{-a^{-1}}$ with $u=a^{-1}-\zeta$ and the root such that $a^{-1} \leq 1 \leq u$, an interesting duality as well as an indication of a double use for the author's first table.} For the record, it may be mentioned that a somewhat different and formally simpler asymptotic formula for differences of the powers at zero has been given by G. Szekeres and F. E. Binet [Ann. Math. Statist. **28** (1957), 494–498; MR **19**, 1169].

J. Riordan (New York)

10962:

Fomenko, O. M. On the distribution with respect to a prime modulus of the products of primes with a given value of a character. Acta Arith. **6** (1960/61), 325–332.

Let $q \equiv 1 \pmod{n}$ be a prime; $w_l^{(s)}$ runs over all products of l different prime factors with $\operatorname{ind} w_l^{(s)} \equiv s \pmod{n}$, where the index is taken with respect to a fixed primitive root of q. Denote by $R_{l,\beta}^{(s)}(N)$ the number of the $w_l^{(s)}$ not exceeding n whose smallest non-negative remainder $\operatorname{mod} q$ is less than βq. Suppose that N is large; put $\log N=r$, let ε_0 and ε_1 be arbitrarily small positive numbers, and assume $\exp r^{\varepsilon_0} \leq q \leq N \exp(-r^{\varepsilon_0})$, $\Delta_1=(1/q+q/N)^{0.5-\varepsilon_1}+N^{-0.2+\varepsilon_1}$. The author proves (1) $R_{l,\beta}^{(s)}(N)=\beta R_{l,1}^{(s)}(N)+O(N\Delta_1)$, which in some sense implies the equidistribution $(\operatorname{mod} q)$ of the indices of the numbers $w_l^{(s)}$. The author obtains (1) as a special case of a theorem dealing with the distribution of the fractional parts of $\alpha w_l^{(s)}$. The author uses the method of Vinogradoff.

P. Erdős (Budapest)

10963:

Linnik, Yu. V. All large numbers are sums of a prime and two squares (A problem of Hardy and Littlewood). I. Mat. Sb. (N.S.) **52** (**94**) (1960), 661–700. (Russian)

If $Q(n)$ is the number of representations of a large number n as a sum of a prime and two squares, the author aims at the inequality (his Theorem 1)

$$(1) \quad Q(n) > .979\pi \frac{n}{\ln n} \prod_p \left(1+\frac{\chi_4(p)}{p(p-1)}\right) \times \prod_{p|n} \frac{(p-1)(p-\chi_4(p))}{p^2-p+\chi_4(p)}$$

(the right-hand side tends to ∞ with n since the reciprocal of the second product is less than $c \ln \ln n$, for a certain absolute positive constant c).

In this paper the author develops part of the heavy machinery required to prove (1) and

$$(2) \quad 5Q(n)+6S(n) = \pi \prod_p \left(1+\frac{\chi_4(p)}{p(p-1)}\right) \times \prod_{p|n} \frac{(p-1)(p-\chi_4(p))}{p^2-p+\chi_4(p)} \left(\frac{5n}{\ln n}+6L(n)\right)+R(n),$$

where $P=\exp(\ln n \ln \ln \ln n/K \ln \ln n)$, K a sufficiently large constant, $L(n)=\sum_{p_1p_2 \leq n;\, p_i>P} 1$, $S(n)$ is the number of solutions of $n=p_1p_2+\xi^2+\eta^2$ with $p_i>P$ $(i=1, 2)$, $|R(n)|<n/(\ln n)^{1+\tau_0}$ for a certain constant τ_0.

S. Chowla (Boulder, Colo.)

10964:

Linnik, Yu. V. All large numbers are sums of a prime and two squares (A problem of Hardy and Littlewood). II. Mat. Sb. (N.S.) **53** (**95**) (1961), 3–38. (Russian)

The author completes the proofs of the results (1) and (2) [#10963]. There is no doubt that the results achieved are of astonishing depth and beauty. It is to be remarked that the Hardy-Littlewood asymptotic formula for the number of representations of a large number as a sum of 4 squares and a prime was first proved (without assuming the Riemann hypothesis) by the reviewer [see Acta Arith. **1** (1935), 115–122]. C. Hooley [Acta Math. **97** (1957), 189–210; MR **19**, 532] first proved the H-L conjecture for the number of representations of a large number as a sum of two squares and a prime, assuming the Riemann hypothesis. (The latter formula is the same as the equation (1) of the preceding review with the "$>$" sign replaced by "$\sim$" (asymptotic equality) and Linnik's .979 by 1.)

S. Chowla (Boulder, Colo.)

10965:

Körner, Otto. Übertragung des Goldbach-Vinogradovschen Satzes auf reell-quadratische Zahlkörper. Math. Ann. **141** (1960), 343–366.

Let K be any real quadratic number field. The author proves an asymptotic formula for the number of representations of a totally positive integer of K as a sum of r prime numbers of K, where $r \geq 3$. This result is analogous to Vinogradov's formula for the number of representations of a natural number as a sum of three primes. Rademacher had previously shown that the author's formula follows from a generalization of the Riemann hypothesis.

T. Estermann (London)

10966:

Lomadse, G. Über die Darstellung der Zahlen durch einige ternäre quadratische Formen. Acta Arith. **6** (1960/61), 225–275.

Exact formulae are obtained, by means of theta functions, for the number of representations of any positive integer in the form $x^2+y^2+7z^2$, $x^2+y^2+10z^2$ and $2x^2+2y^2+5z^2$, where x, y, z are integers.

T. Estermann (London)

10967:

Newman, Donald J. A simplified proof of Waring's conjecture. Michigan Math. J. 7 (1960), 291–295.

The author gives a simple proof of the well-known theorem that every integer is the sum of a bounded number of kth powers of positive integers. He uses the circle method, but only needs crude estimates instead of the usual asymptotic formulae. *P. Erdős* (Budapest)

10968:

Wetzker, Lothar. Eine asymptotische Partitionenformel für Zerfällungen in Elemente gewisser Multiplamengen. Arch. Math. **11** (1960), 259–262.

Let $\mathfrak{T}=\{t_1, \cdots, t_k\}$ be a set of pairwise coprime integers greater than 1; $\mathfrak{M}(\mathfrak{T})$ the set of all positive integers not divisible by any t_i; and $p(n, \mathfrak{M}(\mathfrak{T}))$ the number of partitions of n into elements of $\mathfrak{M}(\mathfrak{T})$. The author proves that, for $n\rightarrow\infty$,

$$p(n, \mathfrak{M}(\mathfrak{T})) \sim 2^{-3/2}\eta^{-1/2}\{\tfrac{2}{3}\psi(\mathfrak{T})\}^{1/4}n^{-3/4} \exp\{\pi(\tfrac{2}{3}\psi(\mathfrak{T})n)^{1/2}\},$$

where $\eta=t_1$ or 1 according as $k=1$ or $k>1$ and $\psi(\mathfrak{T})=\prod_{i=1}^{k}(1-t_i^{-1})$. This result is shown to be an easy consequence of a theorem on partitions due to A. E. Ingham [Ann. of Math. (2) **42** (1941), 1075–1090; MR **3**, 166]; cf. also F. C. Auluck and C. Haselgrove [Proc. Cambridge Philos. Soc. **48** (1952), 566–570; MR **14**, 138]. A special case of the problem considered in the paper under review had been investigated much earlier by K. Knopp and I. Schur [Math. Z. **24** (1925), 559–574], who obtained an asymptotic formula not for the partition function itself but for its logarithm. *L. Mirsky* (Sheffield)

10969:

Grosswald, Emil. Correction and addition to "Some theorems concerning partitions". Trans. Amer. Math. Soc. **95** (1960), 190.

The author remarks that Theorem 2 of the abovementioned paper (that is, formula (2) of the review) [same Trans. **89** (1958), 113–128; MR **20** #3840] is valid only provided that the set $\{a\}$ of least positive residues mod q does not consist of the single element $a=q$, and he points out that in this exceptional case his problem reduces trivially to known results. He confirms by a short argument that this is indeed the only case in which the statement of his Theorem 2 needs modification.

H. Halberstam (London)

10970:

Erdős, P.; Rényi, A. Additive properties of random sequences of positive integers. Acta Arith. **6** (1960), 83–110.

Random sequences of positive integers are defined as follows. Let $p_1, p_2, \cdots$ be given numbers, $0\leq p_i\leq 1$, and let $\xi_1, \xi_2, \cdots$ be a sequence of completely independent random variables. Each ξ_n takes the values 1 and 0 only, with probabilities p_n and $1-p_n$, respectively. Let $\nu_1, \nu_2, \cdots$ be those values of n for which $\xi_n=1$, with $\nu_1<\nu_2<\cdots$. Then $\{\nu_k\}$ is called a random sequence of positive integers, generated by the probability sequence $\{p_k\}$. The authors investigate what happens if special sequences in additive number theory (e.g., squares or primes) are replaced by random sequences with a corresponding order of magnitude. The set of numbers which can be written as the sum of two squares has density zero, but if the random sequence $\{\nu_k\}$ has about the same order of magnitude as the sequence $\{k^2\}$, then the set of all sums $\nu_k+\nu_l$ has positive density (a similar result was obtained by Atkin [unpublished Cambridge dissertation]) and in fact for each r ($r=0, 1, \cdots$) the set of n's having exactly r representations $n=\nu_k+\nu_l$ has positive density, and these densities form a Poisson distribution (all these statements hold with probability 1). And the number $f(n)$ of representations is, in case it tends to infinity, normally distributed in the limit. Similar results are obtained for $\nu_k+\mu_l$, when both $\{\nu_k\}$ and $\{\mu_l\}$ are random sequences, and for $k^2+\mu_l$. Sums of more than two terms of a sequence, and differences of a sequence, are also investigated. If the random sequence $\{\nu_k\}$ is of the order $k^{2+\epsilon}$, then $f(n)$ is almost surely bounded. This provides a proof of the existence of sequences of positive integers $a_1<a_2<\cdots$ such that the number $f(n)$ of representations $n=a_k+a_l$ is bounded, with $a_k=O(k^{2+\delta})$.

The authors also consider Romanoff's theorem which states that the set of all $n=a^k+p$ ($k=1, 2, 3, \cdots$; p prime) has positive density (a is a fixed integer >1). They show that this remains true, with probability 1, if the primes are replaced by a random sequence with probabilities $b/\log(n+1)$. There is also a more general theorem about the sum of a dense and a rare sequence.

N. G. de Bruijn (Eindhoven)

10971:

Niven, Ivan. Uniform distribution of sequences of integers. Trans. Amer. Math. Soc. **98** (1961), 52–61.

The author defines a sequence $\{a_k\}$ of integers to be u.d. (mod m) (uniformly distributed (mod m)) if for every $j=0, 1, \cdots, m-1$, the limiting frequency of indexes k for which $a_k\equiv j$ (mod m) is $1/m$. A sequence A which is u.d. (mod m) for every $m\geqq 2$ is simply said to be u.d. The following are the principal results obtained in this paper. If A is u.d. (mod m) and the complementary sequence $\bar{A}$ has positive asymptotic density, then $\bar{A}$ is also u.d. (mod m). The sequence $\{[k\theta]\}$ is u.d. if and only if θ is irrational or the reciprocal of an integer. If f is a polynomial with integral coefficients, then $\{f(k)\}$ is u.d. if and only if $f(x)=\pm x+c$. Uniform distribution (mod m) implies uniform distribution (mod l) if and only if $l|m$. If A is u.d. (mod m), then

$$(*)\qquad \lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}\exp(2\pi i a_k/m) = 0,$$

but (*) implies that A is u.d. (mod m) only for $m=2, 3$. The validity of (*) for all m does not imply that A is u.d.

W. J. LeVeque (Ann Arbor, Mich.)

10972:

Chowla, S.; Mordell, L. J. Note on the nonvanishing of $L(1)$. Proc. Amer. Math. Soc. **12** (1961), 283–284.

The authors give a simple proof of the fact that

$$\sum_{m=1}^{\infty}\left(\frac{m}{p}\right)\frac{1}{m}\neq 0$$

where p is any odd prime, and $\left(\frac{m}{p}\right)$ is the Legendre symbol. The novelty of the proof consists in an ingenious use of the Gaussian product

$$P=\prod_{n\text{ non-res.}}\left(1-e\left(\frac{n}{p}\right)\right)\Big/\prod_{r\text{ residue}}\left(1-e\left(\frac{r}{p}\right)\right).$$

D. J. Newman (New York)

10973:

Kubota, T. Local relation of Gauss sums. Acta Arith. 6 (1960/61), 285–294.

The Gauss sum $\tau(\chi)$ of a congruence character χ of an algebraic number field F is essentially the same as the constant factor $w(\chi)$ appearing in the functional equation of Hecke's L-function defined by the character. With this interpretation $\tau(\chi)$ is naturally decomposed into its local components (or local Gauss sums) $\tau_{\mathfrak{p}}(\chi)$, where $\mathfrak{p}$ means a finite or infinite place of F. In this paper the author develops a number of important arithmetical properties of the sums $\tau_{\mathfrak{p}}(\chi)$. He considers first the factor set $j_{\mathfrak{p}}(\chi,\ \psi)=\tau_{\mathfrak{p}}(\chi)\tau_{\mathfrak{p}}(\psi)/\tau_{\mathfrak{p}}(\chi,\ \psi)$ between local Gauss sums. Such factor sets ordinarily reduce to Jacobi sums. But, in the general case of local Gauss sums, in particular in the case where the conductors of χ, ψ are divisible by a higher power of $\mathfrak{p}$, there is no simple expression for $j_{\mathfrak{p}}(\chi,\psi)$ as an ordinary Jacobi sum. However, in § 1 the author derives formulas which show that $j_{\mathfrak{p}}(\chi,\psi)$ is in every case transformed into a generalized Jacobi sum.

In general, the problem of explicitly determining the value of $j_{\mathfrak{p}}(\chi,\psi)$ necessarily involves the introduction of the concept of a "Grössencharakter". However, in § 2 the author shows that if χ, ψ are quadratic characters, then the square of the generalized Jacobi sum $j_{\mathfrak{p}}(\chi,\psi)$ is an easily determined natural number, and the sign of $j_{\mathfrak{p}}(\chi,\psi)$ itself is given by the quadratic norm residue symbol. The author's formula is equivalent to a splitting formula for the quadratic norm residue symbol. This splitting formula is, in turn, a local form of the fact that inverse factors such as $(\alpha|\beta)(\beta|\alpha)$ of quadratic residue symbols are expressed by a factor set between Gauss sums. For prime ideals prime to 2, the proof is based upon a simple computation, and for prime ideals dividing 2 it makes use of the product formula of the norm residue symbol and of the analytic properties of L-functions.

A. L. Whiteman (Los Angeles, Calif.)

10974:

Rieger, G. J. Dedekindsche Summen in algebraischen Zahlkörpern. Math. Ann. **141** (1960), 377–383.

Let $B(x)$ be the Bernoulli function defined as 0 if x is an integer and $x-[x]-\frac{1}{2}$ if x is not an integer. For a rational number $x=u/v$ Eisenstein's well-known formula for $B(x)$ may be written in the form

$$\text{(1)}\qquad B(x)=\frac{i}{2v}\sum_{h=1}^{v-1}e^{2\pi ixh}\cot\frac{\pi h}{v}.$$

Also if $(u,v)=1$ the Dedekind sum $D(x)$ may be expressed in terms of $B(x)$ by means of

$$\text{(2)}\qquad D(x)=\sum_{h=1}^{v}B\left(\frac{h}{v}\right)B(xh).$$

In this paper the author develops direct generalizations of (1) and (2) defined over arbitrary algebraic number fields. Moreover he derives a number of basic properties of the extended functions.

A. L. Whiteman (Los Angeles, Calif.)

10975:

Newman, Morris. Subgroups of the modular group and sums of squares. Amer. J. Math. **82** (1960), 761–778.

Put $(\sum_{n=-\infty}^{\infty}q^{n^2})^s=\sum_{n=0}^{\infty}r_s(n)q^n$; also define $r_s(x)=0$ for $x\neq 0, 1, 2, \cdots$. The paper is concerned with relationships of the following kind:

$$r_3(np^2)-r_3(n)=(p-(-n/p))(r_3(n)-r_3(n/p^2)),$$

where p is an odd prime, n an arbitrary integer and $(-n/p)$ the Legendre symbol. The principal purpose of the paper is to prove that formulas of this type exist for all positive integers s and all odd primes p; the number of terms in these formulas depends only on s and p. The method employed is the "subgroup" method which is concerned with functions invariant with respect to a suitable subgroup of the modular group. The reviewer notes that the group-theoretic preliminaries of the paper are developed in greater detail than is required for the present paper; also several theorems concerning the structure of modular subgroups are proved that are not needed in the remainder of the paper. The main results of the paper are too complicated for a brief statement. However we note the following special results:

$$\begin{aligned} r_s(np^2)&=\{1+p^{s-2}-(-1)^{(s-1)(p-1)/4}p^{(s-3)/2}(n/p)\}r_s(n)\\ &\qquad -p^{s-2}r_s(n/p^2) \qquad\qquad (s=1,3,5,7),\\ r_s(np^2)&=\{1+p^{s-2}-(-1)^{(s-1)(p-1)/4}p^{(s-2)/2}(n/p)^2\}r_s(n)\\ &\qquad -p^{s-2}r_s(n/p^2) \qquad\qquad (s=2,4,6,8).\end{aligned}$$

L. Carlitz (Durham, N.C.)

10976:

Касселс, Дж. В. С. [Cassels, J. W. S.]. ★Введение в теорию диофантовых приближений [An introduction to Diophantine approximation]. Translated by A. M. Polosuev; edited and supplemented by A. O. Gel'fond. Izdat. Inostr. Lit., Moscow, 1961. 213 pp. 0.84 r.

For a review of the original [Cambridge Univ. Press, New York, 1957], see MR **19**, 396.

10977:

Roth, K. F. Rational approximations to algebraic numbers. Proc. Internat. Congress Math. 1958, pp. 203–210. Cambridge Univ. Press, New York, 1960.

This is a report of the one-hour lecture which the author delivered at the Edinburgh Congress in 1958. He outlines a proof of his celebrated theorem on the approximation of algebraic numbers by rationals [Mathematika **2** (1955), 1–20; corrigendum, 168; MR **17**, 242] and lists some generalizations and extensions of it. Inter alia, he raises the problem of obtaining an analogous theorem concerning simultaneous approximations to two given algebraic numbers.

C. G. Lekkerkerker (Amsterdam)

10978:

Mikolás, Miklós. On a problem of Hardy and Littlewood in the theory of diophantine approximations. Publ. Math. Debrecen **7** (1960), 158–180.

The Hardy-Littlewood problem referred to is the problem of obtaining estimates for sums of the type $\sum_{j=1}^{N} B_r(n_j x - [n_j x])$ and their quadratic integral means, where $B_r(x)$ is the Bernoulli polynomial of degree r and the n_j are distinct positive integers. The case $r=1$ has been treated by I. S. Gál [Nieuw Arch. Wisk. (2) **23** (1949), 13–38; MR **10**, 355], by making use of the known identity

$$(1) \quad \int_0^1 (au-[au]-\tfrac{1}{2})(bu-[bu]-\tfrac{1}{2})\,du = \frac{(a,b)}{12[a,b]},$$

where (a, b) is the g.c.d. and $[a, b]$ the l.c.m. of a and b. In a previous paper [Acta Sci. Math. Szeged **13** (1949), 93–117; MR **11**, 645] the present author has generalized (1) and indeed proved

$$(2) \quad \int_0^1 \zeta(1-s, au-[au])\zeta(1-s, bu-[bu])\,du = 2(\Gamma(s))^2 \frac{\zeta(2s)}{(2\pi)^{2s}} \left(\frac{(a,b)}{[a,b]}\right)^s \quad (\Re(s) > \tfrac{1}{2}),$$

where $\zeta(s, u)$ is the Hurwitz zeta-function.

The object of the present paper is to further generalize (2) and to study the corresponding generalized Hardy-Littlewood problem. The first main result is the following. Let

$$f_1(u) * f_2(u) \,|\, (x) = \int_0^1 f_1(x-t) f_2(t)\,dt$$

and put $Z_s(u) = (\Gamma(s))^{-1}\zeta(1-s, u-[u])$; also let s_1, s_2 be arbitrary complex numbers, α, β arbitrary integers, $\Lambda = [|\alpha|, |\beta|]$. Then if $x \neq j/\Lambda$ $(j = 0, \pm 1, \pm 2, \cdots)$ and $\Re(s_1) > 0$, $\Re(s_2) > 0$, $\Re(s_1) + \Re(s_2) > 1$, it follows that

$$\begin{aligned} Z_{s_1}(\alpha u) * Z_{s_2}(\beta u) \,|\, (x) &= \frac{|\alpha|^{s_1}|\beta|^{s_2}}{\Lambda^{s_1+s_2}} Z_{s_1+s_2}((\operatorname{sg}\alpha)\Lambda x) \qquad (\operatorname{sg}\alpha = \operatorname{sg}\beta), \\ &= \frac{|\alpha|^{s_1}|\beta|^{s_2}}{\Lambda^{s_1+s_2}\sin \pi(s_1+s_2)} [\sin \pi s_1 Z_{s_1+s_2}((\operatorname{sg}\alpha)\Lambda x) \\ &\qquad + \sin \pi s_2 Z_{s_1+s_2}((\operatorname{sg}\beta)\Lambda x)] \\ &\qquad (\operatorname{sg}\alpha \neq \operatorname{sg}\beta,\ s_1+s_2 \neq \text{integer}). \end{aligned}$$

The remaining results of the paper are concerned with sums of the type $\sum_{j,k=1}^{N} (n_j, n_k)/n_j^{\sigma_1} n_k^{\sigma_2}$ and are of a rather complicated nature. The following special case may, however, be cited. Put $\mathfrak{N}^{\rho}_{\sigma_1,\sigma_2}(N) = \sum_{j,k=1}^{N} (j,k)^{\rho}/j^{\sigma_1}k^{\sigma_2}$. Then

$$\mathfrak{N}^{\rho}_{\sigma_1,\sigma_2}(N) = \frac{\zeta(\sigma_1)\zeta(\sigma_2)}{\zeta(\sigma_1+\sigma_2)} \zeta(\sigma_1+\sigma_2-\rho) + O(N^Q \log N),$$

where $Q = \max(\rho+1-\sigma_1-\sigma_2, 1-\sigma_1)$.

L. Carlitz (Durham, N.C.)

10979:

Andrews, George E. An asymptotic expression for the number of solutions of a general class of Diophantine equations. Trans. Amer. Math. Soc. **99** (1961), 272–277.

Let C be a strictly convex body in real Euclidean n-space with surface content $S(C)$ and N points with integer coordinates on its boundary. It is shown that a function $K(n)$, independent of C, exists such that $S(C) > K(n)N^{(n+1)/n}$. This inequality applied to the case where C is defined as the set of points $(x_1, x_2, \cdots, x_n)$ with $f(x_1, x_2, \cdots, x_n) \leq R$, $f(x_1, x_2, \cdots, x_n)$ being a homogeneous function, yields the inequality $cR^{(n-1)n/(n+1)} > N$, where c is dependent of $f(x_1, x_2, \cdots, x_n)$. This improves the known result that $cR^{n-1} > N$.

D. Derry (Vancouver, B.C.)

10980:

Artyuhov, M. M. Formulas for the number of solutions of some systems of linear diophantine inequalities. Mat. Sb. (N.S.) **51 (93)** (1960), 501–514. (Russian).

Using the methods and results of an earlier note [Dokl. Akad. Nauk SSSR **118** (1958), 215–218; MR **21** #1298] the author obtains formulae for the number of integral solutions of the set of inequalities $m_i \leq f_i \leq M_i$ $(1 \leq i \leq n)$, where the f_i are homogeneous linear forms with integral coefficients in n variables and the m_i and M_i are integers. The formulae are rather elaborate and depend on the minors of the matrix of coefficients of the forms f_i. The reviewer suspects that everything would have been simplified by applying an integral unimodular transformation of the variables so as to bring the matrix of coefficients to triangular form.

J. W. S. Cassels (Cambridge, England)

10981:

Maass, Hans. Über die räumliche Verteilung der Punkte in Gittern mit indefiniter Metrik. Math. Ann. **138** (1959), 287–315.

The author studied the spatial distribution of the points and sub-lattices of lattices with indefinite metric. He [Math. Ann. **137** (1959), 319–327; MR **21** #6359] and others have considered similar questions for Euclidean lattices. Here, however, new tools are needed and these are provided by C. L. Siegel's theory of indefinite quadratic forms [ibid. **124** (1951), 17–54; MR **16**, 800]. The considerations are very involved and the results do not lend themselves to concise quotation. We therefore content ourselves with mentioning a special result about the ternary case. Let S be a rational symmetric matrix of signature (1, 2) and assume that the form $S[\mathfrak{x}]$ does not rationally represent 0. Let $\mathfrak{F}$ be a polygonal fundamental domain corresponding to a group of integral automorphisms of S of finite rank. Let $\mathfrak{B} \subset \mathfrak{F}$ be Riemann measurable and $\mathfrak{B}_0$ be the corresponding domain in $S[\mathfrak{x}] > 0$. Then we have for the number $\alpha_q(S, \mathfrak{B})$ of integral $\mathfrak{g} \in \mathfrak{B}_0$ satisfying $0 < S[\mathfrak{g}] \leq q$ the asymptotic formula

$$\alpha_q(S, \mathfrak{B}) \sim \frac{2\pi}{3} \|S\|^{-1/2} V(\mathfrak{B}) q^{3/2}$$

as $q \to \infty$, where $V(\mathfrak{B})$ denotes the hyperbolic volume of $\mathfrak{B}$.

A. Dvoretzky (Jerusalem)

10982:

Вальфиш, А. З. [Val'fiš, A. Z.]. ★Целые точки в многомерных шарах [Integral points in many-dimensional spheres]. Akad. Nauk Gruzin. SSR, Mat. Inst. A. M. Razmadze, Izdat. Akad Nauk Gruzin. SSR, Tbilisi, 1959. 460 pp. 24 r.

This is almost a literal translation into Russian of the author's book *Gitterpunkte in mehrdimensionalen Kugeln*, Państwowe Wydawnictwo Naukowe, Warsaw, 1957 [MR

20 #3826]. Like the German text, this book is written in perfect "Landau style". There are also improvements in the mathematical contents; for example, the theorem $P_4(x)=Bx\log^{3/4}x(\log\log x)^{1/2}$ on page 93 of the German edition appears in the following sharpened form in the Russian edition (also on p. 93): $P_4(x)=Bx\log^{2/3}x$ [note $\frac{2}{3}<\frac{3}{4}$].
S. Chowla (Boulder, Colo.)

FIELDS

See also 11006, 11241, 11242.

10983:

Nakano, Noboru. Über die Produkte und Quotienten von Idealen in unendlichen algebraischen Zahlkörpern. J. Sci. Hiroshima Univ. Ser. A **19** (1955/56), 239–253.

Continuing his earlier work [Proc. Internat. Sympos. Algebraic Number Theory (Tokyo & Nikko, 1955), pp. 249–251, Science Council of Japan, Tokyo, 1956; MR **19**, 396; and further references in the review cited], the author derives necessary and sufficient conditions that $\mathfrak{a}=\mathfrak{b}\mathfrak{a}$ or $\mathfrak{a}:\mathfrak{b}=\mathfrak{a}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in the ring of integral elements of an infinite algebraic extension of the rational field. The conditions depend on the primary components of $\mathfrak{a}$ and $\mathfrak{b}$ and are valid in any algebraic number field.
G. Whaples (Bloomington, Ind.)

10984:

Vaida, Dragoş. Sur les corps partiellement ordonnés. Com. Acad. R. P. Romîne **9** (1959), 1243–1248. (Romanian. Russian and French summaries)

The author discusses properties of the order relation in a not-necessarily-commutative field sufficient to imply such conditions as commutativity and the Archimedean property.
M. M. Day (Urbana, Ill.)

10985:

Ribenboim, P. Remarques sur le prolongement des valuations de Krull. Rend. Circ. Mat. Palermo (2) **8** (1959), 152–159.

Hasse showed [Math. Ann. **95** (1925), 229–238] that for algebraic number fields K and an integer $n=\sum_{i=1}^{g} e_i f_i$, there exists an algebraic extension $\bar{K}$ of degree n over K such that a prime ideal p of K may be factored in $\bar{K}$ as $\prod_{i=1}^{g} P_i^{e_i}$, where the prime ideals P_i have the relative degrees f_i. In this note the author shows, among other results, how Hasse's existence theorem may be extended to fields K admitting suitable discrete valuations of finite rank. Sufficient conditions for the existence of mutually independent prolongations are also derived.
O. F. G. Schilling (Chicago, Ill.)

ABSTRACT ALGEBRAIC GEOMETRY

See also 11006.

10986:

Ohm, Jack. Semiprojective completions of abstract curves. Illinois J. Math. **5** (1961), 345–350.

Let T be a birational correspondence between two algebraic varieties V and V', all defined over a field k. The union of k-loci of points P' on V' which correspond non-biregularly to points on V under T is called the non-biregular locus of T on V' and is denoted by $\mathcal{N}_T'$. The set of points P' on V' such that T is not complete over P' is called the pseudo-point locus of T on V' and is denoted by $\mathcal{P}_T'$. These subsets are k-closed, and the latter one is treated in the case where T is just an algebraic correspondence.

An extension V^* of an abstract variety V is called semi-projective if V^* is expressed as the union of V with a subset V'' of a birationally equivalent projective variety V' in such a way that $V''=V'-\mathcal{N}_T'$, where T is the birational correspondence which defines the identification of points of V and V''.

The main result of the article is now stated as follows: Let U^r be a subvariety of a variety V, let k be a field of definition for U and V, let $\bar{V}$ be the projective join of the projectively embedded representatives of V and let $\bar{T}$ be the natural correspondence between V and $\bar{V}$. If U corresponds biregularly under $\bar{T}$ to a subvariety $\bar{U}$ of $\bar{V}$, then there exist a projectively embeddable variety V' and a birational correspondence T between V and V' such that (i) both T and V' are defined over k, (ii) U corresponds biregularly to a variety U' (under T) which is k-isomorphic to the projective variety $\bar{U}$, and (iii) $\mathcal{N}_T'\cap\mathcal{P}_T'\cap U'$ is either empty or has dimension $\leq r-2$.

An immediate consequence of the result is that if U is a curve on a variety V, then there is a semi-projective extension V' of V such that U corresponds to a projective curve on V'.

One should note that this last result implies that if an abstract variety V is not complete, then there is an abstract variety V' which contains V as a proper open subset.
M. Nagata (Kyoto)

10987:

Abe, Ei-ichi. Dualité de Tannaka des groupes algébriques. Tôhoku Math. J. (2) **12** (1960), 327–332.

One half of this paper is devoted to a fancy interpretation of a result by the reviewer [(1) of Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 43–50; MR **17**, 897], stating that an irreducible group-variety possesses a maximal linear representation (i.e., a homomorphism onto an algebraic group of matrices which enjoys the universal mapping property). The main result of this part would state, in ordinary language, that such a maximal representation is an isomorphism if and only if the given group-variety is isomorphic to an algebraic group of matrices. The other half of this paper is an exercise on a similar theme: let G be an irreducible group-variety over an algebraically closed field k; let H be a group-subvariety of G; set $V=G/H$ (right G-homogeneous space), and consider $k(V)$ to be canonically imbedded in $k(G)$; then the group of the automorphisms of $k(V)$ over k which are G-admissible is isomorphic, as a group, to N/H, N being the normalizer of H in G.
I. Barsotti (Providence, R.I.)

10988:

Koizumi, Shoji. On Albanese varieties. Illinois J. Math. **4** (1960), 358–366.

Sia V una varietà irriducibile, proiettiva, senza singolarità di dimensione massima, sul corpo algebricamente chiuso k; sia f l'applicazione razionale canonica di V

sulla propria varietà A di Severi-Albanese; sia V_n il prodotto simmetrico di V per se stessa n volte (più precisamente, sia V_n la varietà rappresentativa del sistema algebrico irriducibile di tutti i cicli effettivi di dimensione 0 ed ordine n su V); sia F_n l'applicazione razionale di V_n su A definita da: $F_nX = \sum_{i=1}^n fX_i$ (la $\sum$ intesa rispetto all'operazione su A) se X è il punto rappresentativo su V_n del ciclo $\sum_{i=1}^n X_i$, $X_i \in V$ (la $\sum$ intesa come somma di cicli). I principali risultati dimostrati sono i seguenti: (1) Per n elevato, l'inseparabilità di F_n è 1; (2) Per n elevato, F_n non ha punti fondamentali su A, ed $F_n[P]$ è irriducibile per ogni $P \in A$ (questa irriducibilità era già stata dimostrata da Y. Taniyama [Sci. Papers Coll. Gen. Ed. Univ. Tokyo 8 (1958), 123–137; MR **21** #4958] ed A. Mattuck [Illinois J. Math. **3** (1959), 145–149; MR **21** #1974]; (3) Per n elevato, $F_n[P]$ è regolare per ogni $P \in A$ (regolare nel senso che la sua varietà di Severi-Albanese si riduce ad un punto).

Se un ciclo di dimensione 0 su V viene detto regolarmente equivalente a 0 quando è differenza di due elementi di un sistema algebrico irriducibile a varietà rappresentativa regolare, i risultati (2) e (3) comportano che l'equivalenza regolare è effettivamente un'equivalenza (ossia i cicli equivalenti a 0 formano un gruppo), e che A è isomorfa, come gruppo, al gruppo residuo del gruppo dei cicli di dimensione 0 su V, modulo il gruppo di quelli regolarmente equivalenti a 0. *I. Barsotti* (Providence, R.I.)

10989:

Koizumi, Shoji. On Albanese varieties. II. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1960), 383–390.

Questo è una continuazione del lavoro recensito sopra; sposta lo studio da $F_n[P]$ (varietà) ad $F_n\{P\}$ (ciclo), e dimostra che quest'ultimo è irriducibile, ossia che $F_n\{P\} = 1F_n[P]$, valido per n elevato e per tutti i $P \in A$. In tal modo A diviene la varietà rappresentativa di un sistema algebrico irriducibile di cicli di dimensione $nd-q$ su $V \times \cdots \times V$ (n volte); qui, $d = \dim V$, $q = \dim A =$ irregolarità superficiale di V.

I. Barsotti (Providence, R.I.)

LINEAR ALGEBRA

See also 11085, 11269a-b.

10990:

Wilson, Edwin Bidwell. ★Vector analysis. A text-book for the use of students of mathematics and physics, founded upon the lectures of J. Willard Gibbs. Dover Publications, Inc., New York, 1960. xviii+436 pp. $2.00.

Unaltered republication of the second edition [Scribner, New York, 1909]. Contents (in brief): Addition and scalar multiplication; direct and skew products; vector differential and integral calculus; linear vector functions (dyadics); rotations and strains.

10991:

Blumenthal, Bernhard. ★Einführung in die Matrizenrechnung: Allgemeinverständliche Darstellung für Nichtmathematiker. VEB Verlag Technik, Berlin, 1960. 47 pp. DM 4.80.

The pamphlet is an introduction for non-mathematicians to the use of matrices. *G. Kron* (Schenectady, N.Y.)

10992:

Paige, Lowell, J.; Swift, J. Dean. ★Elements of linear algebra. Ginn and Co., Boston, Mass., 1961. xvi+348 pp. $7.00.

This text is designed as a first course, primarily for students of mathematics and physics. It reads easily and gives sufficient details and examples to make the transition from problem-type courses to algebraic theory a reasonable one. Extreme abstraction is avoided, but reasonable rigor is employed to develop an appreciation of modern standards.

After an introductory chapter containing the terminology of sets and mappings, Chapter 2 presents three-dimensional real vectors with emphasis on geometric applications to provide motivation for the more general material to follow. Many of the important techniques and concepts are introduced here and discussed in relation to the geometry of euclidean three-space.

The vector space $V_n(R)$, with vectors represented as n-tuples of real numbers, is presented in Chapter 3. While all discussion of vector spaces is limited to finite-dimensional real spaces, the attempt is made to develop the subject in such a way that generalization will be relatively easy.

A short chapter on euclidean n-space from the point of view of coordinate systems is included in order to strengthen the geometric interpretation of $V_n(R)$. Determinants are introduced as functions from sets of n vectors in $V_n(R)$ into the real numbers, and the elementary properties are derived using this definition.

Linear transformations and algebraic operations on them are discussed, and from these a discussion of matrices and matrix properties arises in a natural way. Groups are presented to describe sets of linear transformations of special types and their associated matrices. Bilinear and quadratic forms are treated with geometric applications.

In Chapter 9, complex numbers and polynomial rings are introduced with discussion of factorization. The solution of polynomial equations is mentioned briefly.

Final chapters include treatment of characteristic values and vectors and similarity of matrices with discussion of canonical forms.

The treatment of the general case of linear equations seems inadequate, being confined to a single short section concerning applications of rank. The restriction to real, finite-dimensional vector spaces will seem an unnecessary limitation to some. However, on the whole, the book is well written and should provide a useful elementary introduction to the subject.

J. E. Whitesitt (Bozeman, Mont.)

10993:

Bouteloup, Jacques. ★Calcul matriciel. "Que sais-je?" Le Point des Connaissances Actuelles, No. 927. Presses Universitaires de France, Paris, 1961. 128 pp.

A pocket-size matrix book which nevertheless contains much introductory information. It uses modern concepts and at the same time includes computational tricks. It goes up to material on diagonalisable matrices and Jordan normal forms, on commuting matrices, on infinite series

of matrices, systems of linear differential equations, and connections with the tensor calculus.

O. Taussky-Todd (Pasadena, Calif.)

10994:

Taussky, Olga. Matrices of rational integers. Bull. Amer. Math. Soc. **66** (1960), 327–345.

This is mainly a literature review covering many aspects of matrices whose elements are rational integers. This is a vast subject, as many parts of mathematics (e.g., parts of combinatorial analysis, number theory, ideal theory, group theory) can be brought under this heading. The bibliography contains about 150 items, arranged into the following chapters: (1a) Combinatorial problems; (1b) The modular group; (1c) Quadratic forms; (1d) Space groups and crystallography; (2) Elementary number theory for matrices; (3) Number theory in hypercomplex systems; (4) Ideal theory in algebraic number fields and linear algebras via matrix theory; (5) Matrix roots of polynomial equations; (6a) Integral group matrices; (6b) Integral group representations. The main emphasis is on the Chapters 5 and 6. *N. G. de Bruijn* (Eindhoven)

10995:

Newman, D. J. Another proof of the minimax theorem. Proc. Amer. Math. Soc. **11** (1960), 692–693.

The minimax theorem, which is the subject of the paper under review, was originally proved by J. von Neumann in his fundamental paper on the theory of games [Math. Ann. **100** (1928), 295–320] with the aid of Brouwer's fixed-point theorem. Since then, research in this field has progressed in two directions. On the one hand, following the work of J. Ville in 1938, purely (or almost) algebraic proofs of ever greater simplicity have been constructed for the minimax theorem. On the other hand, the theorem has been made the point of departure for a number of far-reaching generalizations. The present paper offers a strikingly short and elegant proof of the original minimax theorem. The principal steps are as follows. Let E^n be the n-dimensional euclidean space. If $x=(x_1, \cdots, x_n)$ and $x_i \geqq 0$ $(i=1, \cdots, n)$, we write $x \geqq 0$. If, in addition, $x \neq 0$, we write $x \geq 0$. M denotes a fixed real $n \times m$ matrix and J the $n \times m$ matrix all of whose elements are equal to 1.

(i) Stiemke's theorem is established, to the effect that, if S is a subspace of E^N and $S^\perp$ is its orthogonal complement, then $S \cup S^\perp$ contains some vector x with $x \geq 0$.

(ii) From (i) the theorem of the alternative is obtained: either $Mx \geq 0$ for some $x \in E^m$ with $x \geq 0$ or $-M^Ty \geq 0$ for some $y \in E^n$ with $y \geq 0$.

(iii) Finally, the minimax theorem is derived in the following form. There exist a real number ν and vectors $x \in E^m$, $y \in E^n$ such that $x \geq 0$, $y \geq 0$ and

$$(M-\nu J)x \geqq 0, \quad (-M^T+\nu J^T)y \geqq 0.$$

{Reviewer's remark. There are a number of misprints. In particular '$m \times n$' is printed in place of '$n \times m$'; 'projection' in place of 'orthogonal projection'; and (most troublesome of all) there is much confusion between the symbols '$\geqq$' and '$\geq$'.} *L. Mirsky* (Sheffield)

10996:

Nikaidô, Hukukane. On a method of proof for the minimax theorem. Proc. Amer. Math. Soc. **10** (1959), 205–212.

The minimax theorem that is proved in this paper is the following: Let X and Y be convex sets which are compact with respect to certain given Hausdorff topologies. For fixed $x, u \in X$, $y, v \in Y$, it is assumed that the mappings $t \to (1-t)x+tu: [0, 1] \to X$ and $t \to (1-t)y+tv: [0, 1] \to Y$ are continuous in these topologies. Let $f(x, y)$ be a real-valued function defined on $X \times Y$ and satisfying: (1) For any fixed $y \in Y$, $f(x, y)$ is a continuous concave function of x on X. (2) For any fixed $x \in X$, $f(x, y)$ is a continuous convex function of y on Y. Then there exists at least one $(\bar{x}, \bar{y}) \in X \times Y$ such that $f(x, \bar{y}) \leqq f(\bar{x}, \bar{y}) \leqq f(\bar{x}, y)$ for all $(x, y) \in X \times Y$.

The proof parallels very closely the Brown-von Neumann differential equations [*Contributions to the theory of games*, Ann. of Math. Studies No. 24, Princeton Univ. Press, 1950; pp. 73–79]. The method is also applied to prove the duality in linear programming.

H. W. Kuhn (Princeton, N.J.)

10997:

Brauer, Alfred; Mewborn, A. C. The greatest distance between two characteristic roots of a matrix. Duke Math. J. **26** (1959), 653–661.

In this paper bounds are obtained for the spread $s(A) = \max|\omega_i - \omega_j|$ of an n-square matrix with eigenvalues ω_i. Set $K(A) = |2(1-1/n)c_1^2 - 4c_2|^{1/2}$, where c_1, c_2 are the first and second elementary symmetric functions of the ω_i, respectively. The authors show that for arbitrary complex A, $s(A) \geqq (2/n)^{1/2}K(A)$ when n is even, and $s(A) \geqq \{2n/(n^2-1)\}^{1/2}K(A)$ when n is odd. The proof is a generalization of a theorem on algebraic equations due to Popoviciu [Mathematika (Cluj) **9** (1935), 129–145]. For a normal matrix A, $K(A)$ may be replaced by $K(B)$, with n replaced by k, where B is any principal minor of order $k \geqq 3$ of A.

For a matrix C with real roots, $s(C) \leqq K(C)$. A simple proof of this result of v. Sz.-Nagy [Jber. Deutsch. Math. Verein. **27** (1918), 37–43] and Mirsky [Mathematika **3** (1956), 127–130; MR **18**, 460]. More generally, upper and lower bounds for $s(C)$ are given in terms of the first $2s$ elementary symmetric functions of the ω_i, $2 \leqq s \leqq [n/2]$.

Among the applications of these results is the following. If A is a generalized stochastic symmetric matrix with row sum $\omega_1 > K(A)$, then A is non-singular.

B. N. Moyls (Vancouver, B.C.)

10998:

Marcus, M.; Greiner, P. On the unitary completion of a matrix. Illinois J. Math. **5** (1961), 152–158.

By a 'diagonal' of an $n \times n$ array is meant a set of positions $(i, \sigma i)$, $1 \leqq i \leqq n$, where σ is a permutation of the symbols 1, 2, $\cdots$, n. Two diagonals will be said to be 'k-overlapping' if they have precisely k positions in common. Let $1 \leqq k \leqq n$ and consider a fixed pair of k-overlapping diagonals on an $n \times n$ array. Denote by s_k the maximum, taken over all $n \times n$ unitary matrices, of the absolute value of the sum of the elements in the given pair of diagonals. The authors then deduce a series of estimates for s_k; for example $s_k \leqq n$ if $n = k+2$ and $s_k \leqq n-4+2 \cot \pi/8$ if $n = k+4$. Moreover, it is shown that, for each of the bounds so deduced, there exists a pair of k-overlapping diagonals and a unitary matrix for which the absolute value of the sum of the elements in these

diagonals is the appropriate bound. In addition, if $0 \leq \mu \leq s_k$, then there exists a unitary matrix such that the absolute value of the sum of the elements in the fixed pair of diagonals is μ. *L. Mirsky* (Sheffield)

10999:

Haynsworth, Emilie V. Bounds for determinants with positive diagonals. Trans. Amer. Math. Soc. **96** (1960), 395–399.

If the real numbers a_{ij} are such that $a_{ii} > 0$, $a_{ii} \geq \sum_{j \neq i} |a_{ij}|$ (i.e., the matrix $A = (a_{ij})$ has weakly dominant main diagonal), a bound is obtained for $\det A$; this is similar to, but more exact than, the bounds of Price [Proc. Amer. Math. Soc. **2** (1951), 497–502; MR **12**, 793], Ostrowski [ibid. **3** (1952), 26–30; MR **14**, 611] and Brenner [ibid. **8** (1957), 532–534; MR **19**, 115], for matrices with nonreal elements. As an application of her general theorem [Duke Math. J. **24** (1957), 313–320; MR **19**, 628] the author also obtains bounds for determinants of matrices which satisfy $a_{ii} \geq nA_i^+ - \sum_{j \neq i} a_{ij}$, $2A_i^+ = \max_{j \neq i} a_{ij} + |\max_{j \neq i} a_{ij}|$, $i = 1, \cdots, n$.

J. L. Brenner (Palo Alto, Calif.)

11000:

Mařík, Jan; Pták, Vlastimil. Norms, spectra and combinatorial properties of matrices. Czechoslovak Math. J. **10** (85) (1960), 181–196. (Russian summary)

It is shown how certain concepts and facts studied so far mainly for non-negative matrices can be employed in a general context. The method is combinatorial and uses graph-theoretical ideas: instead of linking points with sets a correspondence between sets and sets is employed. The main matrix result concerns matrices $A = (a_{ik})$ of complex elements for which $|A| = \max_i \sum_k |a_{ik}| \leq 1$. These matrices will either have all their characteristic roots inside the unit circle, in which case $|A^p| < 1$ for some p, or will satisfy $|A^p| = 1$ for all p. In the latter case A has a characteristic root on the unit circle. It is shown that $q(n) = n^2 - n + 1$ has the property that $|A^{q(n)}| = 1$ implies $|A^r| = 1$ for all r. The companion matrix of $x^n + \frac{1}{2}(x^{n-1} - x^{n-2})$ shows that this result is best possible. If A is reducible, $n^2 - 3n + 3$ plays the same role as $q(n)$, while if all a_{ii} are $\neq 0$, the corresponding number is n. More detailed results can be described after the concepts of reducible, irreducible and primitive matrices have been introduced. These concepts have been much used in the study of non-negative matrices. They are used here for arbitrary complex matrices.

The following combinatorial concepts are introduced using the distribution of the zeros in the matrix. An $n \times n$ matrix A generates the following mapping $\varphi = F(A)$ of the set of subsets of the set N of the integers $1, 2, \cdots, n$ into itself: for any $S \subset N$ let $\varphi(S)$ be the set of those $j \in N$ for which there exists an $i \in S$ such that $a_{ij} \neq 0$. These special mappings are elements of the set $\mathcal{F}$ of all mappings of subsets of N having $\varphi(0) = 0$ and such that (1) $\varphi(S) \subset N$ if $S \subset N$ and (2) $\varphi(S_1 \cup S_2) = \varphi(S_1) \cup \varphi(S_2)$ for any two sets $S_1, S_2 \subset N$. Composition of the φ's is defined as superposition. A mapping $\varphi \in \mathcal{F}$ is reducible if either $\varphi(N) = 0$ or there exists an $S \neq 0$, N such that $\varphi(S) \subset S$. A mapping which is not reducible is irreducible. A mapping $\varphi \in \mathcal{F}$ is primitive if it is irreducible and has the property that no disjoint system of sets $R, \varphi(R), \cdots, \varphi^{k-1}(R)$ exists with union N and such that $\varphi^k(R) = R$. The following three conditions are equivalent: φ^k is irreducible for $k = 1, \cdots, n$, φ is primitive, φ^k is irreducible for all k. The theorem that the $[(n-1)^2+1]$th power of a primitive non-negative matrix has every element $\neq 0$ is generalized to mappings. This fact is finally used to obtain the main theorem of this paper. It is further shown that the counterexample showing that $(n-1)^2+1$ is best possible is essentially unique.

The special mapping $\varphi = F(A)$ generated by a matrix A has the property that reducibility, irreducibility, primitivity for φ imply the same for A. For non-negative matrices A, B the mapping $F(A)$ has the property $F = (AB)$ $F(A)F(B)$, in particular $F(A^r) = F(A)^r$. This latter fact is now extended to the case of matrices A with $|A| \leq 1$ and all $\sum_k |a_{ik}^{(r)}| = 1$ for some r, where $A^r = (a_{ik}^{(r)})$. A vital tool in the proofs is the study of $P(x)$, the set of $i \in N$ for which $|x_i| = 1$ in a vector $x = x_1, \cdots, x_n$ with $\max |x_i| \leq 1$, and of $P(A)$, the set of those $i \in N$ for which $\sum_j |a_{ij}| = 1$. They are related in the case $|A| \leq 1$ and $\max |x_i| \leq 1$ by: $P(x) \supset \varphi(P(Ax))$. It is shown, e.g., that $|\alpha| = 1$ is possible for an irreducible A with $|A| \leq 1$ if and only if $A = \alpha DMD^{-1}$, where D is the diagonal matrix formed by the components of the characteristic vector of α (which are shown to be of absolute value 1), and where $m_{ij} = |a_{ij}|$; further $\sum_j |a_{ij}| = 1$ for all i. (That non-negative irreducible A with $|A| \leq 1$ and $\alpha = 1$ are stochastic was known previously.) Further, denote by A_V, for a given A, the matrix with elements a_{ik}, $i, k \in V \subset N$. Assume $|A| \leq 1$, and let $T_1, \cdots, T_r$ be the set of all nonempty minimal $T \subset N$ such that $\varphi(T) \subset T$ for $\varphi = F(A)$. Then each characteristic root α with $|\alpha| = 1$ is a root of some A_{T_i}. The paper contains a great many further results and techniques. [See also Pták, same J. **8** (**83**) (1958), 487–495; MR **22** #1520a; Pták and Sedláček, ibid. 496–501; MR **22** #1520b].

{Since the absolute value of the largest characteristic root is majorized by any norm, the main question of this paper could be asked with respect to some other norm.}

O. Taussky-Todd (Pasadena, Calif.)

11001:

Bkouche, Rudolphe. Sur une classe d'endomorphismes d'un espace vectoriel hermitien de dimension finie. C. R. Acad. Sci. Paris **251** (1960), 2851–2853.

Let K be a commutative field with involutory automorphism J, E a finite-dimensional vector space over K, and h a non-degenerate Hermitian form on E. The author defines an ortho-endomorphism of E as an endomorphism f satisfying this condition: if a subspace V of E is invariant under f, so is its orthogonal space V^+. His study of ortho-endomorphisms is confined mainly to the case where K is the quadratic extension of a maximal ordered field, $J \neq 1$, and the index ν of f is 0 or 1. For $\nu = 0$, an endomorphism f is an ortho-endomorphism if and only if it is normal. For $\nu = 1$, an ortho-endomorphism f is diagonalizable if and only if one of the following holds: (a) f has an eigenvector of negative length; (b) f has two non-collinear eigenvectors of zero length. *G. E. Wall* (Sydney)

11002:

Noël, Guy. Transformations pseudo-hermitiennes d'un vectoriel pseudo-unitaire. Acad. Roy. Belg. Bull. Cl. Sci. (5) **46** (1960), 834–854.

This paper contains a classification of pseudo-hermitian

transformations in a finite-dimensional pseudo-unitary space with respect to the pseudo-unitary changes of coordinates. The prefix 'pseudo' indicates that the scalar product is not positive definite. If relative to a pseudo-orthonormal basis the scalar product of two vectors is

$$\langle x, y\rangle = \sum_{r=1}^{p} x^r \bar{y}^r - \sum_{s=1}^{q} x^{p+s}\bar{y}^{p+s}$$

(i.e., has p plus and q minus signs), then certain statements can be made about the elementary divisors of a pseudo-hermitian transformation, the following being typical: the transformation possesses at least $|p-q|$ real characteristic roots, and never has an elementary divisor relative to a real characteristic root of exponent greater than $2\min(p, q)+1$. Canonical forms are obtained for pseudo-hermitian matrices, and the paper concludes with a discussion of the modifications that have to be made in the theory when the space is (real) pseudo-euclidean, the transformations being pseudo-symmetric and the co-ordinate changes pseudo-orthogonal. *H. S. Ruse* (Leeds)

11003:

Spencer, A. J. M. The invariants of six symmetric 3×3 matrices. Arch. Rational Mech. Anal. **7** (1961), 64–77.

This paper supplements an earlier one [same Arch. **4** (1960), 214–230; MR **22** #715] by proving that the 6! invariants obtained by permuting the factors in trace **abcdef** (where **a**, $\cdots$, **f** are symmetric 3×3 matrices) can be expressed linearly in terms of ten of them.

J. A. Todd (Cambridge, England)

11004:

Rinehart, R. F. Skew matrices as square roots. Amer. Math. Monthly **67** (1960), 157–161.

If H is a negative definite, or semidefinite, hermitian matrix of rank r, it is proved that H possesses matrix square roots, that every normal square root is skew-hermitian and that every square root of rank r is similar to a skew-hermitian square root of H. The real analogues are also given. *H. S. A. Potter* (Aberdeen)

ASSOCIATIVE RINGS AND ALGEBRAS

See also 11047.

11005:

Джекобсон, Н. [Jacobson, Nathan]. ★Строение колец [Structure of rings]. Translated from the English by V. A. Andrunakievič; edited with a foreword by A. G. Kuroš. Izdat. Inostr. Lit., Moscow, 1961. 392 pp. 1.80 r.

The original [Amer. Math. Soc., Providence, R.I., 1956] was reviewed in MR **18**, 373.

11006:

Zariski, Oscar; Samuel, Pierre. ★Commutative algebra. Vol. II. The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto-London-New York, 1960. x+414 pp. $7.75.

[For vol. I, 1957, see MR **19**, 833.] It would be an understatement to say that the second volume of this work lives up to the standards and expectations set by the first, because the scope and style of the final volume could not have been anticipated even though the first volume contained an outline of the authors' total program and ample evidence of their expository skill. Unlike the first, however, the second volume is concerned in large measure with those parts of commutative algebra that are the fruits of its union with algebraic geometry; and this fact has had a marked influence on the organization of the material and the character of the exposition. Throughout the work the algebro-geometric motivations and applications of the purely algebraic material are elaborated in such detail as to make the book (among other things) an excellent text for a course in the arithmetic foundations of algebraic geometry. This fact alone renders the book a welcome addition to the all too sparse list of expository treatises in this area. Yet it is more than this, for much of the material appears here in book form for the first time, many of the proofs are new, and the appendices in particular contain some heretofore unpublished results.

There are three chapters and seven appendices. The first (Chapter VI) deals with valuation theory, the second contains a detailed study of polynomial and power-series rings, while the last is concerned with local algebra. No attempt has been made to include a discussion of homological algebra, but the influence of that discipline is evident at several points, notably in the sections on characteristic functions and chains of syzygies. In the opinion of the reviewer the work would have gained somewhat by the inclusion of a more extended discussion of M. Nagata's treatment of algebraic geometry over Dedekind domains, and an account of D. Rees' studies on pseudo-valuations, since these topics fall naturally into the context of the book and constitute natural and important adjuncts to the material treated there. However, the book is not intended to be exhaustive, and such omissions are to be preferred to the sacrifice of the clarity and detail which are distinguishing assets of the work and make it an invaluable instrument for both pedagogy and research. The brief summary of its contents that we give below will give some indication of the scope of the work.

The discussion of valuations in Chapter VI is preceded by a study of places. The valuation ring of a place is introduced together with the notions of dimension, rank, and specialization, and the existence of places, the characterization of integrally closed domains as intersections of valuation rings, and the behavior of places under field extensions are among the topics studied before the value group is mentioned. Moreover, an indication of the role played by the notion of a place in algebraic geometry is discussed here in a separate section. Deeper aspects of the theory are examined after the notion of the value group has been introduced and valuations have been fully defined. These include the questions (including those of existence) related to the rank, rational rank and dimension, approximation theorems, a detailed discussion of ramification theory, Krull rings, and prime divisors. A long section on the abstract Riemann surface of a field includes a treatment of topological properties of such surfaces and those of models of the field. The chapter closes with a section on derived normal models for varieties defined over restricted domains in the sense of M. Nagata. However, the normalization lemma stated at the bottom of page 124 appears to be in error except when

the ground ring R is a field, so that the treatment of the material in this section must be confined to this case.

After preliminary sections on power-series rings and graded rings, Chapter VII contains five sections devoted to varieties in the affine and projective space. The Nullstellensatz is proved, the relation between affine and projective varieties is examined, and properties of arithmetically normal varieties are considered. Four sections on dimension theory in finite integral domains, polynomial rings and power-series rings are followed by a section on the behavior of polynomial ideals under ground-field extension. The chapter closes with the two long sections on characteristic functions and chains of syzygies mentioned above.

Chapter VIII opens with a discussion of topological rings and their completions, topological modules and Zariski rings. A section on Hensel's lemma and its applications is followed by three sections devoted to characteristic functions, dimension theory, and the theory of multiplicities. The sections on regular local rings and the Cohen structure theorem include a proof of the unique-factorization theorem for equicharacteristic regular local rings that is based on a lemma communicated by M. Nagata. {A proof of this theorem for arbitrary regular local rings due to the junior author and based on properties of chains of syzygies is given in Appendix 7.} Chapter VIII closes with a discussion of the analytical irreducibility and normality of normal varieties.

The first five appendices are due to the senior author while the junior author wrote the sixth and seventh. The first deals with the relationship between prime ideals in a noetherian domain $\mathfrak{o}$ and those in a simple ring extension $\mathfrak{o}'=\mathfrak{o}[t]$ of $\mathfrak{o}$. It is concerned primarily with the inequality, $h(\mathfrak{p}')+\dim_{\mathfrak{o}/\mathfrak{p}}\mathfrak{o}'/\mathfrak{p}'\leq h(\mathfrak{p})+\dim_{\mathfrak{o}}\mathfrak{o}'$, where $\mathfrak{p}$ and $\mathfrak{p}'$ are prime ideals such that $\mathfrak{p}'\cap\mathfrak{o}=\mathfrak{p}$ and h denotes the height (or rank) of the ideal. The second deals with valuations in noetherian domains and is concerned with extensions to quotient fields of arbitrary noetherian domains of results concerning rank, rational rank and dimension that are known for algebraic function fields. These extensions are due originally to S. Abhyankar, but new proofs that do not use the Cohen structure theorems are given here. Appendix 3 is concerned with valuation ideals in integral domains, and in Appendix 4 properties of the integral closure of modules over an integral domain are studied. This material has interesting applications to the theory of linear systems with assigned base points, and these applications are discussed here. Appendix 5 constitutes a new treatment of Zariski's work on complete ideals in polynomial rings that is valid in arbitrary 2-dimensional regular local rings. While having wider validity, the methodology here is essentially simpler than in the earlier work. Appendix 6 is concerned with Macaulay rings. It contains simple non-homological proofs of some results proved originally by D. Rees with the aid of homological methods, as well as some new results on such rings. As has already been mentioned, Appendix 7 is devoted to the proof of the fact that every regular local ring is a unique factorization domain. *H. T. Muhly* (Iowa City, Iowa)

11007:

Faith, Carl. Radical extensions of rings. Proc. Amer. Math. Soc. **12** (1961), 274–283.

A ring A is called a radical extension of a subring B if for every $a\in A$ some power of a belongs to B. The author continues his investigations in determining the rings which are radical extensions over proper subrings [I. Kaplansky, Canad. J. Math. **3** (1951), 290–292; MR **13**, 101; C. Faith, Proc. Amer. Math. Soc. **11** (1960), 43–45; MR **22** #2636] and obtains some conclusive remarks in this direction: Let A be radical over a proper subring B, then: (1) if A is a ring with no non-zero nil ideals, then A is a field; (2) if A is semi-simple (in the sense of Jacobson) and B is commutative, then A is also commutative; (3) if B is a division ring, then A is necessarily of the form $Q\oplus N$, where N is an arbitrary nil ideal and Q is a directly irreducible radical extension of B. Finally, a necessary and sufficient condition that Q is a directly irreducible radical extension of a proper division subring B is that the identity e of B is also the identity of Q, and that the Jacobson radical J of Q is nil and Q/J a field of prime characteristic which is a radical extension of $(B+J)/J$. Consequently, if B is a division ring, which is non-commutative or of characteristic zero, then its radical extensions are of the form $B\oplus N$, where N is an arbitrary nil ring. The author also discusses certain properties of the ring A which are inherited by its subring B of which A is a radical extension: If A is primitive, then B is also primitive and actually B is a dense ring of linear transformations of the same space V over which A is a dense ring. If A is semi-simple, then B is also semi-simple. In the converse, he proves that if B is simple and A is not a radical ring and has no non-zero nil ideals then A is primitive; furthermore, if B is simple with an identity, then A is also simple with the same identity. The paper also contains some results on rings satisfying the relation $x^ny^m-y^mx^n=0$ with m, n depending on x and y. *S. A. Amitsur* (Jerusalem)

11008:

Posner, Edward C.; Schneider, Hans. Hyperplanes and prime rings. Arch. Math. **11** (1960), 322–326.

Let R be a prime ring with a minimal left ideal; then R is also primitive, and hence isomorphic to a ring of linear transformations of a right vector space F over a division ring D which are continuous with respect to a fixed dual space E; R also contains all linear transformations of finite rank. The authors prove that if D contains less than r elements then there are $r+1$ elements c_i in R such that the product $c_1ac_2a\cdots ac_{r+1}=0$ for all $a\in R$. Conversely, if D contains at least r elements, then for every set $c_1,\cdots,c_{r+1}$ of $r+1$ elements in R there exists $a\in R$ such that $c_1ac_2a\cdots ac_{r+1}\neq0$. This gives a partial answer to the question: under what conditions on a prime ring R does the validity of $c_1ac_2a\cdots ac_{r+1}=0$ for all $a\in R$ imply that some $c_i=0$? (The case $r=1$ is the defining property of primeness.) An interesting feature of the proof is the use of a notion of a covering of a vector space by linear functionals. {α_1 on p. 325, line 11, should be α_i.}
S. A. Amitsur (Jerusalem)

11009:

Huzurbazar, M. S. The multiplicative group of a division ring. Dokl. Akad. Nauk SSSR **131** (1960), 1268–1271 (Russian); translated as Soviet Math. Dokl. **1**, 433–435.

Let K be an associative noncommutative division ring, K^* the totality of nonzero elements of K, and Z the center of K, $Z^*=Z\cap K^*$. L. K. Hua [Acad. Sinica Sci. Record **3**

(1950), 1–6; MR **12**, 584] proved that K^*/Z^* is not finitely solvable, and has no center; W. R. Scott [Proc. Amer. Math. Soc. 8 (1957), 303–305; MR **18**, 788] extended this by showing that K^*/Z^* has no abelian normal subgroups $\neq 1$. The author's generalization states that K^*/Z^* contains no locally nilpotent normal subgroup $\neq 1$. The proof depends on four lemmas, the first of which is essentially Hua's Lemma 1 [loc. cit.]; the proof of the first part of the second lemma is patterned after Scott's first lemma [loc. cit.]; the third lemma is a trivial statement, while the fourth lemma is the statement of Scott's theorem (which is assumed here, and which does not follow immediately from the Cartan-Brauer-Hua theorem as the author indicates). {Some misprints occur toward the end of the proof of Lemma 2 in the translation. Specifically, λ (not Λ, as is printed) is a root of unity, so it is a primitive lth root of unity, say. Then norm $f(\lambda)$ is defined (not norm $f(x)$, as is printed). In the last sentence of the proof, read $b_q{}^{k_m}(\lambda-1)^{N_m}$ in place of $b_q{}^k m(\lambda-1)^N m$.} Scott [loc. cit., Th. 3] showed that $A \cap B \subseteqq Z^*$, for normal subgroups A, B of K^*, implies that either $A \subseteqq Z^*$ or $B \subseteqq Z^*$. The author applies his theorem to sharpen this as follows: $[A, B] \subseteqq Z^*$ implies $A \subseteqq Z^*$ or $B \subseteqq Z^*$. *Carl Faith* (Princeton, N.J.)

11010:

Johnson, R. E. Ideal extensions in an algebra. Proc. Cambridge Philos. Soc. **57** (1961), 427–428.

A linear associative algebra R without unit over a field may be embedded in an algebra R' having the unit of the base field as unit. Smithies [same Proc. **55** (1959), 277–281; MR **22** #1825] gave conditions under which a right [left, two-sided] ideal A of R may be embedded in a unique right [left, two-sided] ideal A' of R' such that $A' \cap R = A$. The purpose of this note is to relate Smithies' results to an earlier paper of the author's [Duke Math. J. **20** (1953), 569–573; MR **15**, 391]. A right ideal A of R was there called weakly prime if whenever $rR \subset A$ then $r \in A$. The main result is then that a regular right ideal A has a unique extension A' of the type described above if and only if A is weakly prime. Regular here means that there exists an element u in R with $(1-u)R \subset A$. The author then shows how most of the results of Smithies' paper [loc. cit.] may be deduced from this theorem.

A. Rosenberg (Ithaca, N.Y.)

11011:

Feller, E. H.; Swokowski, E. W. Reflective N-prime rings with the ascending chain condition. Trans. Amer. Math. Soc. **99** (1961), 264–271.

The authors define a ring R satisfying the ascending chain condition on right and left ideals to be N-prime if $R/N(R)$, $N(R)$ the nil-radical of R, is a prime ring, and reflective if an element of R is regular if and only if its image in $R/N(R)$ is regular. They prove then that R is a reflective N-prime ring if and only if it has a right and left quotient ring $Q(R)$ which is a complete matric ring over a completely primary ring and $Q(R/N(R)) \simeq Q(R)/N(Q(R))$. This generalizes a result of A. W. Goldie [Proc. London Math. Soc. (3) **8** (1958), 589–608; MR **21** #1988] on prime rings to certain rings with radical. Also the polynomial ring $R[x]$ of a reflective N-prime [prime] ring R is shown to be reflective N-prime [prime].

W. E. Deskins (E. Lansing, Mich.)

11012:

Nöbauer, Wilfried. Über die Ableitungen der Vollideale. Math. Z. **75** (1960/61), 14–21.

In the present paper the author continues the study of the derivatives of full ideals initiated in two earlier papers [Monatsh. Math. **64** (1960), 176–183, 335–348; MR **22** #5652, 8037]. In addition to some general theorems on derivatives and D-kernels of full ideals, the author also computes the D-kernels of the Restpolynomideale of a finite field and of the Restpolynomideale of the rational integers. For the last topic he makes use of a theorem of E. G. Straus [Proc. Amer. Math. Soc. **2** (1951), 24–27; MR **12**, 700] on polynomials with rational coefficients all of whose derivatives are integral at the integers.

L. Carlitz (Durham, N.C.)

11013:

Feller, Edmund H. Intersection irreducible ideals of a non-commutative principal ideal domain. Canad. J. Math. **12** (1960), 592–596.

Für durchschnitts-irreduzible ein- und zweiseitige Ideale in nichtkommutativen Hauptidealringen werden einige Eigenschaften angegeben, die an bekannte Resultate von H. Fitting [Math. Ann. **111** (1935), 19–41] und O. Teichmüller [S.-B. Preuß. Akad. Wiss. Berlin Phys.-Math. Kl. **1937**, 169–177] anschließen; hierbei wird auf eine frühere Arbeit des Verf. [Trans. Amer. Math. Soc. **81** (1956), 342–357; MR **17**, 1047] sowie auf N. Jacobson, *The theory of rings* [Amer. Math. Soc., New York, 1943; MR **5**, 31] verwiesen. Z. B. gilt für das Radikal P eines irreduziblen Rechtsideals aR in seinem Eigenring (Normalisator) B die Darstellung $P = \{aR|b\}_l = \{x | xb \in aR,\ x \in R\}$, wo bR das eindeutig bestimmte minimale Oberideal von aR ist; B/P ist ein Schiefkörper. Ein zweiseitiges Ideal $a^*R = Ra^*$ ist irreduzibel genau dann, wenn $a^* = u(p^*)^n$ für eine ganze Zahl $n \geqq 0$, wo u eine Einheit und p^* ein Primelement von R mit $p^*R = Rp^*$ ist. Anwendung auf den Polynomring $Q[x]$, wo Q den Quaternionenschiefkörper über einem reell-abgeschlossenen Körper bezeichnet.

H.-J. Hoehnke (Berlin)

11014:

Kovács, L. G.; Szép, J. Rings covered by minimal left ideals. Publ. Math. Debrecen **7** (1960), 194–197.

The authors prove that a ring R is the set-theoretic union of its minimal left ideals if and only if it is either a zero-ring whose additive group is an elementary p-group or a ring with left unit e such that Re is a field which, together with the ideal of left annihilators of R, generates R. The covering of an R-module G by minimal submodules is resolved by considering the elements of the Dorroh-extension R^* of R which annihilate certain elements of G, and the results are used to prove the main theorem. (R^* is the ring of all pairs $[r, n]$, where $r \in R$ and n is a rational integer, addition being defined componentwise and multiplication by the equation $[r, n][r', n'] = [rr' + nr' + n'r, nn']$.)

W. E. Deskins (E. Lansing, Mich.)

11015:

Samuel, Pierre. Algèbre locale et ensembles analytiques. Séminaire P. Lelong, 1957/58, exp. 2, 12 pp. Faculté des Sciences de Paris, 1959.

The author presents some well-known results on local

rings, in particular on rings of convergent power series in several variables, such as Weierstrass preparation theorem, unique-factorization theorem, etc. *M. Nagata* (Kyoto)

11016:

Walter, John H. Structure of cleft rings. II. Illinois J. Math. **4** (1960), 376–396.

Let $R=S\oplus N$ and $R'=S'\oplus N'$ be cleft rings and let $I_0:S\to S'$ be an isomorphism. The conditions for the extendability of I_0 to an isomorphism $I:R\to R'$, given by the author in the first paper of this series [same J. **3** (1959), 445–467; MR **21** #7233], are here recast in homological terms. Let T_{ji} be the submodule of elements of H_{ji} which vanish on S, and let $I_i:S_i\to S_i'$ be the isomorphism induced by I_0 in the simple components S_i of S $(i,j=1,\cdots,k)$. Then I_0 is extendable to an isomorphism $I:R\to R'$ if and only if there are (I_j,I_i)-isomorphisms $\theta_{ji}:T_{ji}\to T_{ji}'$ which commute with the coboundary operator $(i,j=1,\cdots,k)$. Using the earlier conditions as well as some structural properties of H_{ji} developed in this paper, the author shows that every automorphism of S can be extended to an automorphism of R. Application of the new criterion is made to direct decompositions $R=S\oplus M\oplus M^2\oplus\cdots\oplus M^r$, where S is a semisimple subring of R, M is an (S,S)-submodule of R, and $N=M\oplus\cdots\oplus M^r$ is the radical of R. Such decompositions are called gradings of R. If $R=S'\oplus M'\oplus\cdots\oplus M'^r$ is a second grading of R and $I_0:S\to S'$ is an isomorphism, then there is an automorphism I of R which extends I_0 and maps M onto M'. This result is then generalized by replacing the minimum condition on R by (1) the minimum condition on R/N^q $(q=1,2,\cdots)$, (2) $\bigcap(N^q;q=1,2,\cdots)=0$, and (3) the completeness of R in the topology defined by the subbase $\{N^q\}$ $(q=1,2,\cdots)$ of neighborhoods of zero. Here a grading of R takes the form $R=S\oplus M\oplus M^2\oplus\cdots\oplus M^r\oplus N^{r+1}$ $(r=1,2,\cdots)$, with S a semisimple subring of R and M an (S,S)-submodule of N. It is shown that an isomorphism of the semisimple components of two such gradings can be extended to an automorphism of R which maps M onto M'. *M. F. Smiley* (Riverside, Calif.)

11017:

Chase, Stephen U. Direct products of modules. Trans. Amer. Math. Soc. **97** (1960), 457–473.

Analysis of conditions of the following type on a ring R: Every direct product of flat right R-modules is flat (if and only if every finitely generated left ideal I in R is finitely related; i.e., if $I=F/K$ with F free and finitely generated, then also K is finitely generated). Every direct product of projective right R-modules is projective (if and only if every finitely generated left ideal is finitely related and R satisfies the minimum condition on principal left ideals; if R is commutative, this is equivalent to the minimum condition on ideals).

The second theorem is actually a consequence of a theorem asserting that a right R-module cannot be both a direct product of ζ copies of R (with $\zeta\geq\operatorname{card} R$) and a pure submodule of a direct sum of modules each of which has $\leq\zeta$ generators, unless R satisfies the minimum condition on principal left ideals. This also gives a partial answer to a problem of Köthe: If every right R-module is a direct sum of finitely generated modules, then R satisfies the minimum condition on right ideals, and every indecomposable injective right R-module has finite length.

These results are also used to retrieve Hattori's characterizations of Prüfer rings and to characterize Dedekind rings by the condition that the torsion submodule of any module is a direct summand if it has bounded order. *D. Zelinsky* (Evanston, Ill.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

11018:

Minc, H. The free commutative entropic logarithmetic. Proc. Roy. Soc. Edinburgh Sect. A **65**, 177–192 (1959).

If $\mathfrak{A}$ is a free cyclic nonassociative groupoid generated by x, each element of $\mathfrak{A}$ can be written in the form x^P, where P is an "index". Under the definitions $x^{P+Q}=x^Px^Q$ and $x^{PQ}=(x^P)^Q$, the indices form an algebraic system $\mathfrak{L}$ in which the additive groupoid is isomorphic to $\mathfrak{A}$ and multiplication is associative and right-distributive. In a previous paper [same Proc. **64** (1957), 319–341; MR **19**, 836] the author has represented elements of $\mathfrak{L}$ diagrammatically as trees and hence deduced faithful representations of $\mathfrak{L}$ by polynomials in $\mathfrak{M}[\lambda,\mu]$, where $\mathfrak{M}$ is the ring of integers and λ, μ are noncommuting indeterminates. For definitions of trees, corresponding index polynomials and other terms not defined here, see the paper cited. The congruence relations $(c)P+Q\sim Q+P$ and $(e)(P+Q)+(R+S)\sim(P+R)+(Q+S)$ determine homomorphisms c and e on $\mathfrak{L}$. The homomorphic image $\mathfrak{L}_{ce}$ is called the free commutative entropic logarithmetic. If two indices, or the corresponding trees, P and Q are congruent mod $(c)(e)$ they are said to be concordant. Two trees are concordant if and only if they have the same number of free ends at each altitude. From this it is shown that $\mathfrak{L}_{ce}$ can be faithfully represented by index polynomials in one indeterminate in which the degree of each term corresponds to the altitude and its coefficient to the number of free ends at this altitude. The final section is devoted to the enumeration of indices and a formula is found for the number of nonconcordant indices of a given altitude. *D. C. Murdoch* (Vancouver, B.C.)

11019:

Minc, H. Enumeration of indices of given altitude and potency. Proc. Edinburgh Math. Soc. (2) **11** (1958/59), 207–209.

"Indices" in the title refers to elements of the free logarithmetic L, which may be regarded as indices of powers of an element x of a general nonassociative algebra A. They are the same as the nonassociative numbers of T. Evans [Amer. Math. Monthly **64** (1957), 299–309; MR **20** #58] and the reviewer's binary partitive numbers [Proc. Roy. Soc. Edinburgh Sect. A **62** (1949), 442–453; MR **10**, 677; cf. ibid. **59** (1939), 153–162]. If A is commutative we have similarly "commutative indices", elements of the free commutative logarithmetic L_c. In either case the indices can also be represented as bifurcating root-trees, respectively with or without the distinction between left and right. The altitude of an index means the altitude of the corresponding tree, while its potency (or degree) means the number of factors x in the power, or of free knots of the tree. Recurrence equations are found for

$p(\alpha, \delta)$, the number of indices of altitude α and potency δ, and for the corresponding number $q(\alpha, \delta)$ of commutative indices, and these numbers are tabulated as far as $\alpha=4$.

I. M. H. Etherington (Edinburgh)

11020:

Minc, H. A problem in partitions: Enumeration of elements of a given degree in the free commutative entropic cyclic groupoid. Proc. Edinburgh Math. Soc. (2) **11** (1958/59), 223–224.

Compare #11019. If the algebra A satisfies both the commutative and entropic laws $xy=yx$, $xy\cdot zw=xz\cdot yw$, and if we identify the indices of powers which are equal in consequence of these laws, we obtain the free commutative entropic logarithmetic which the author denoted by $\mathfrak{L}_{ce}$ in #11018. He now solves the problem of enumerating the indices in $\mathfrak{L}_{ce}$ of degree (or potency) $\delta=d+1$ by connecting it with the following problem. Given two positive integers c, d, let $v(c, d)$ be the number of partitions of d into positive integers of form $d=c+c_1+c_2+\cdots+c_n$, where $c_1\leqq 2c$, $c_{i+1}\leqq 2c_i$. Since $v(c, d)=\sum_{i=1}^{2c} v(i, d-c)$, values of $v(c, d)$ can be calculated recursively. They are tabulated as far as $c=8$, $d=14$, and the first row $(c=1)$ gives the numbers of indices of degree $d+1$.

I. M. H. Etherington (Edinburgh)

11021:

Urbanik, Kazimierz; Wright, Fred B. Absolute-valued algebras. Proc. Amer. Math. Soc. **11** (1960), 861–866.

The authors prove that an absolute-valued algebra with a unit over the field of real numbers is isomorphic to either the real field, the complex field, the quaternions, or the Cayley-Dickson algebra. They show by a set of examples that the existence of a unit element is essential for the validity of the theorem. The proof consists in showing that the algebra in question must necessarily be algebraic. Thus, reduction to Albert's result [Ann. of Math. (2) **48** (1947), 495–501; MR **8**, 561] is made.

O. F. G. Schilling (Chicago, Ill.)

HOMOLOGICAL ALGEBRA

See also 11016, 11017, 11027a–b, 11068.

11022:

Fröhlich, A. On groups over a d.g. near-ring. I. Sum constructions and free R-groups. Quart. J. Math. Oxford Ser. (2) **11** (1960), 193–210.

A near-ring differs from a ring in that addition need not be commutative, and that multiplication need not be left-distributive. It is distributively generated (d.g.) if those elements, multiplication by which is left-distributive, generate the additive group. A d.g. near-ring is a natural operator domain for a nonabelian group. For example, the set of maps of a group into itself generated by the endomorphisms under pointwise multiplication is a d.g. near-ring.

If R is a d.g. near-ring, the category of R-groups and R-homomorphisms is shown to possess free and cartesian sums. These are perhaps better known [cf. Grothendieck, Tôhoku Math. J. (2) **9** (1957), 119–221; MR **21** #1328] as sums and products. Further, the existence of "sufficiently many" free R-groups is demonstrated. The notion of a projective R-group is defined in the usual way, and it is shown that these are precisely the retracts of free R-groups.

A. Heller (Urbana, Ill.)

11023:

Fröhlich, A. On groups over a d.g. near-ring. II. Categories and functors. Quart. J. Math. Oxford Ser. (2) **11** (1960), 211–228.

The notion of additive category is generalized to the nonabelian case, the maps from one object to another forming a (not necessarily abelian) group; composition is right- but not in general left-distributive. For R-groups one example (the "left category") is given by the set $\mathrm{Hom}_R(\Omega, \Delta)$ of maps generated by the R-homomorphisms under pointwise multiplication. Another (the "right category") is defined, relative to a suitably chosen set of generators of Ω, by requiring distributivity with respect to the generators.

Relative to such a structure the notion of an additive functor is defined. $\mathrm{Hom}_R(\Omega, \Delta)$ is an additive functor of Δ into a left category of groups, and of Ω into a right category.

An application is given to the notion of a variety of groups: a "variety" in a category of R-groups is a set of objects closed under image, co-image and product. These are shown to coincide with the kernels of certain additive functors. The notion may also be dualized to that of a covariety.

A theory of derived functors is promised for a later paper.

A. Heller (Urbana, Ill.)

11024:

Norguet, François. Notions sur la théorie des catégories et l'algèbre homologique. Séminaire P. Lelong, 1957/58, exp. 9, 21 pp. Faculté des Sciences de Paris, 1959.

Résumé of the elements of the theory of categories from the point of view of homological algebra, as a background to the author's further lectures in this Séminaire [#11410, 11411 below].

GROUPS AND GENERALIZATIONS

See also 10994, 11018.

11025:

Piccard, Sophie. Les groupes fondamentaux et leur décomposition en produit quasi libre. C. R. Acad. Sci. Paris **251** (1960), 2450–2452.

The subset A of a group G is called a basis of G if it generates G, but ceases to do so if any finite number of its elements are replaced by a smaller number of elements of G. If each of a family of subgroups G_λ of G has a basis and if, for all choices of bases A_λ of G_λ, $\bigcup A_\lambda$ is a basis of G, then G is described as the quasi-free product of the subgroups G_λ. Thirteen propositions are stated without proof. Most of them follow immediately from the definitions, but Proposition 2 calls for some justification.

P. J. Higgins (London)

11026:

Baumslag, Gilbert. Wreath products and finitely presented groups. Math. Z. **75** (1960/61), 22–28.

The main theorem proved in this paper states that the (restricted) wreath product of two finitely presented groups A and B is finitely presented if and only if either A is trivial or B is finite. The proof makes use of an ingeniously constructed product of A and B that is intermediate between their wreath and free products. A corollary of the theorem is the effective undecidability of the question whether a member of a recursive class of finitely generated and recursively presented groups can be finitely presented.

B. H. Neumann (Manchester)

11027a:

MacHenry, Trueman. The tensor product and the 2nd nilpotent product of groups. Math. Z. **73** (1960), 134–145.

11027b:

MacHenry, Trueman. The tensor product of non-abelian groups and exact sequences. Arch. Math. **11** (1960), 166–170.

In the first paper, the author presents a proof, in current terminology, of an old theorem of H. Whitney [Duke Math. J. **4** (1938), 495–528; Theorem 11]: Let A and B be groups with commutator subgroups A' and B'. Then $A \otimes B \cong (A/A') \otimes (B/B')$. He then goes on to show that $A \otimes B \cong [A, B]^{(2)}$, the metabelian commutator subgroup of Golovin [Mat. Sb. (N.S.) **28 (70)** (1951), 431–444; Amer. Math. Soc. Transl. (2) **2** (1956), 117–131; MR **13**, 105; **17**, 824], herein renamed the second nilpotent commutator subgroup of A and B. In the second paper, the author extends, from the abelian case to the non-abelian case, the result stating that $\otimes G$ is a right exact functor. He introduces the right exact functor which carries a group G onto its abelianization G/G'. Let C be a group presented as F/R where F is free. In the abelian case, it is known that Tor $(C, G) = \ker (R \otimes G \to F \otimes G)$. The author shows that, in the general case, Tor $(C/C', G/G')$ is a direct summand of the above kernel, so that tensor products of non-abelian groups have at least one property which is obscured upon abelianization.

F. Haimo (St. Louis, Mo.)

11028:

Struik, Ruth Rebekka. On nilpotent products of cyclic groups. Canad. J. Math. **12** (1960), 447–462.

The author extends results on free nilpotent groups to nilpotent products of cyclic groups. For nilpotent products, see Golovin [Mat. Sb. (N.S.) **27 (69)** (1950), 427–454; Amer. Math. Soc. Transl. (2) **2** (1956), 117–132; MR **12**, 672; **17**, 824]. For a second nilpotent product of three cyclic groups of odd order, every element is shown to have a unique expansion as a product of standard commutators [M. Hall, Proc. Amer. Math. Soc. **1** (1950), 575–581; MR **12**, 388], where the factors are chosen from fourteen such commutators arranged by non-decreasing weight with each weight ≤ 3, but are otherwise arbitrary. The theorem is generalized to the case of any finite number of cyclic groups of odd or of zero order. The result can be further extended to higher nilpotent products if the primes which divide the orders of the finite cyclic groups involved are large. The case of orders which are each a power of two is more complicated, even for second nilpotent products, and one can no longer make a nearly arbitrary choice of the factors in the expansion.

F. Haimo (St. Louis, Mo.)

11029:

Cohen, Eckford. On the normal number of irreducible factors of a finite abelian group. Ricerche Mat. **9** (1960), 203–212.

Let X denote the semigroup of the finite abelian groups with respect to the direct product. Let $\omega(G)$ be the number of irreducible factors of a group G contained in X. $\rho(G)$ denotes the order of G. The author shows

$$\sum_{\rho(G)\leq x} \omega(G) = \alpha x \log \log x + \beta x + O(x/\log x).$$

Further, if for any $\delta > 0$, $N_\delta(x)$ denotes the number of $G \subset X$ such that $\rho(G) \leq x$ and $|\omega(G) - \log \log x| > (\log \log x)^{1/2+\delta}$, then $N_\delta(x) = o(x)$. Similar results hold for $\Omega(G)$, the number of direct factors of G which are powers of irreducible groups of X. The proofs are based on a theorem of Erdös and Szekeres [Acta Litt. Sci. Szeged **7** (1934), 95–102] on the number of abelian groups of a given order.

H.-E. Richert (Göttingen)

11030a:

Žuravskiĭ, V. S. On the splitting of certain mixed abelian groups. Mat. Sb. (N.S.) **48 (90)** (1959), 499–508. (Russian)

11030b:

Žuravskiĭ, V. S. Generalization of some criteria for splitting of mixed abelian groups. Mat. Sb. (N.S.) **51 (93)** (1960), 377–382. (Russian)

For a mixed abelian group G, consider the exact sequence $0 \to F \to G \xrightarrow{\phi} H \to 0$, where F is the torsion subgroup of G, and where we assume that the aperiodic group H is reduced. Let $\chi(g)$, for $g \in G$, be the set of all natural numbers which divide g in G (i.e., there exists, for each such natural number n, at least one $x \in G$ with $nx = g$); this is called the characteristic of g in G. In general, $\chi\phi(g) \supset \chi(g)$, where the first characteristic is computed with respect to H; but a result of Liapin's [Mat. Sb. (N.S.) **8 (50)** (1940), 205–237; MR **3**, 195] shows that, if F splits G, there always exists an $f_g \in F$ with $\chi\phi(g) = \chi(g + f_g)$. One says that $g + f_g$ is a maximal element in the coset $g + F$. Conversely, if each coset of F possesses a maximal element, does F split G? No, but conditions not too restrictive can be appended to the requirement of maximal elements in each coset to obtain splitting criteria: if, for instance, each primary component F_p of F has no non-trivial elements of infinite height at p, and if H is p-complete for each prime p for which $F_p \neq 0$, then the maximal element condition is enough for F to split G. If H is the direct sum of subgroups H_α and if $K_\alpha = \phi^{-1}(H_\alpha)$, the complete inverse image, then F splits G if and only if F splits each K_α. If H is a group of rank 1 (therefore, as an aperiodic group, representable as a subgroup of the additive group of the rationals with 1 as an element), and if F_p is without elements of infinite height at p (except 0) for each prime p which divides infinitely into that element of H which corresponds to 1, then the maximal element condition suffices for F to split G.

In the second paper, the author forms the subgroup $F_p^{(1)}$ of all $x \in F_p$ of infinite height at p in F_p. If H is of rank 1 or if $F_p = 0$ for all primes p for which H is not p-complete, then the maximal-element condition suffices for $G/\Sigma \oplus F_p^{(1)}$ to be split by its periodic part $F/\Sigma \oplus F_p^{(1)}$, where the direct sum is taken over all p for which H is

p-complete. If F is p-primary, if H is p-complete of rank 1 and if the maximal element condition holds, then a necessary and sufficient condition for F to split G is the existence of a uniform bound on the orders of the elements of $F_p^{(1)}$. If H is the direct sum of subgroups H_a of rank 1, or p-complete for each p such that $F_p \neq 0$, if each $\Sigma_{(a)} \oplus F_p^{(1)}$ has uniformly bounded orders for its elements, where summation is taken over all p for which H_a is p-complete, and if the maximal element condition holds, then F splits G. *F. Haimo* (St. Louis, Mo.)

11031:

Heineken, Hermann. Eine Verallgemeinerung des Subnormalteilerbegriffs. Arch. Math. **11** (1960), 244–252.

The author considers the following generalization of subnormality in a group G. A subgroup A is subnormal in a broad sense (written $A \triangleleft\triangleleft\triangleleft G$) if there exists an ascending tower θ of subgroups of a group G with the following properties: (I) G and A are contained in θ; (II) if two different subgroups C and D are contained in θ, then either $C<D$ or $D<C$; (III) if C and D are contained in θ, and $C<D$, then there exists a proper normal subgroup N of D such that $C \leq N$; (IV) for every set of subgroups contained in θ, the intersection is contained in θ. A group G is called a (V)-group if $A \triangleleft\triangleleft\triangleleft G$ and $N \triangleleft G$ imply that $AN \triangleleft\triangleleft\triangleleft G$. A group G is called a V^*-group if $A \triangleleft\triangleleft\triangleleft G$, $B \triangleleft\triangleleft\triangleleft G$, $C \triangleleft\triangleleft\triangleleft \{A, B\}$ and $N \triangleleft \{A, B\}$ imply that $CN \triangleleft\triangleleft\triangleleft \{A, B\}$.

With these definitions, and using routine techniques, the author is able to produce meager generalizations of Wielandt's results on subnormality. The following theorem is typical: If G is a V^*-group, $A \triangleleft\triangleleft\triangleleft G$ and $B \triangleleft\triangleleft\triangleleft G$, A is perfect and there exists no proper normal subgroup N of A with $N(A \cap B^A) = A$, then A will normalize B.
L. J. Paige (Los Angeles, Calif.)

11032:

Wiegold, James. On a note of B. H. Neumann. J. London Math. Soc. **35** (1960), 63–64.

Necessary and sufficient conditions for a group to carry a mean in the sense of R. Schimmack [Math. Ann. **68** (1910), 125–132] were given by W. R. Scott [Amer. J. Math. **74** (1952), 667–675; MR **13**, 910] and the reviewer [J. London Math. Soc. **28** (1953), 472–476; MR **15**, 99]. The author shows that they are equivalent to the condition that both the centre of the group and its factor group are direct products of groups isomorphic to the additive group of rational numbers. *B. H. Neumann* (Manchester)

11033:

Newman, M. F. On a class of nilpotent groups. Proc. London Math. Soc. (3) **10** (1960), 365–375.

A metabelian group G is here called "just metabelian" if every proper factor group is abelian. If the centre Z is non-trivial, then G must be nilpotent of class 2 (i.e., the commutator group G' is contained in Z) and in this case G is called "just nilpotent-of-class-2" or briefly a $JN2$-group. If G is such a group, then G' is the intersection of all non-trivial normal subgroups and so is of order p (a prime), while Z is cyclic of order p^n (some positive integer n) or of type p^∞. These conditions, together with $G' \leq Z$, are also sufficient to ensure that G is a $JN2$-group.

A further simple property of $JN2$-groups is that the group modulo the centre is an elementary p-group. Thus G/Z is a vector space over $GF(p)$ and the mapping: $aZ, bZ \to f(a, b)$, where $[a, b]$ is the $f(a, b)$th power of a fixed generator of G', is a non-singular alternating form on G/Z. If the space G/Z possesses a symplectic basis, then the classification of $JN2$-groups can be carried further. Specifically, the $JN2$-group G is called an $SJN2$-group if G contains a subset S whose cosets form a basis of G/Z and such that every $s \in S$ commutes with all but a countable number of elements of S. If G is an $SJN2$-group, then G/Z does possess symplectic bases and of course all countable $JN2$-groups are of this type. However, an example shows that not every $JN2$-group is an $SJN2$-group.

The second half of the paper contains a complete classification of $SJN2$-groups, in the sense that each $SJN2$-group yields a quartet of invariants and the set of these quartets is in one-one correspondence with the isomorphism classes of $SJN2$-groups.
K. Gruenberg (London)

11034:

Benado, Mihail. Über den Kommutatrixbegriff. Proc. London Math. Soc. (3) **10** (1960), 514–530.

In an earlier note [C. R. Acad. Sci. Paris **244** (1957), 1595–1597; MR **19**, 385], the author introduced functions q which generalize group commutators on those lattices which generalize the lattices of operator subgroups of a group. Suppose that X and Y are operator subgroups of a non-commutative group G. Then, with mild restrictions on q, it turns out that $q(X, Y)$ is the intersection of the commutator subgroup $[X, Y]$ with a verbal subgroup in the case where the subgroup generated by X and Y is a regular product of X and Y in the sense of Golovin [Mat. Sb. (N.S.) **27** (**69**) (1950), 427–454; Amer. Math. Soc. Transl. (2) **2** (1956), 89–115; MR **12**, 672; **17**, 824]. A generalization is then made to more than two variables X and Y. Closely related is a series of papers by S. Moran [Proc. London Math. Soc. (3) **6** (1956), 581–596; **8** (1958), 548–568; **9** (1959), 287–317; MR **20** #3908, 7054; **21** #4178]. *F. Haimo* (St. Louis, Mo.)

11035:

Yacoub, K. R. A note on the classes of non-linear semi-special permutations. Proc. Glasgow Math. Assoc. **4**, 3–6 (1958).

Semi-special permutations π on the integers from 1 to n have been extensively studied by the author. For instance, see same Proc. **2** (1955), 116–123; **3** (1958), 164–169; **3** (1956), 18–35 [MR **17**, 11; **20** #5235c; **19**, 5]. Let $n = p^\alpha$, where p is an odd prime and where $\alpha > 2$. Let s be a principal number for π (see third paper cited above for definition). In the second paper cited above the author distinguishes two classes of non-linear semi-special permutations. In the present case he is able to show that these classes are non-empty and that one is included in the other. *F. Haimo* (St. Louis, Mo.)

11036:

Weichsel, Paul M. On a theorem of Iwasawa. Proc. Amer. Math. Soc. **12** (1961), 148–150.

Let G be a finite group. A subgroup H of G is called a p-complement of G (p prime) if the index of H in G is

equal to the order of a p-Sylow subgroup of G. The author gives the following characterizations of finite groups which are abelian, Hamiltonian, or nilpotent:

1. Let the order of a group G be divisible by more than two distinct primes. Then G is abelian (Hamiltonian, or nilpotent) if and only if for every prime p dividing the order of G, G contains a p-complement which is abelian (Hamiltonian, or nilpotent).

2. A group G of order $g=\prod_k p_k^{\alpha_k}(\alpha_k>0)$ is abelian (Hamiltonian, or nilpotent) if and only if G contains subgroups of order $p_i^{\alpha_i}p_j^{\alpha_j}$ for all i, j $(i\neq j)$ and these subgroups are abelian (Hamiltonian, or nilpotent).

K. Iwasawa (Cambridge, Mass.)

11037:

Feit, Walter. On groups which contain Frobenius groups as subgroups. Proc. Sympos. Pure Math., Vol. 1, pp. 22–28. American Mathematical Society, Providence, R.I., 1959.

The author studies the following situation. Let G be a finite group and M a subgroup such that: (1) no element of M except the identity commutes with an element outside M; (2) the meet of any two distinct conjugates of M is the identity; (3) $M\neq N\neq G$, where N is the normalizer of M. (1)–(3) imply that N is a Frobenius group with regular subgroup M and thus that M is nilpotent. Suppose that M has $k+1$ irreducible characters and write $m=|M|$, $q=|N:M|$. The main technical result, Theorem 1, states that, if (4) $q\neq m-1$ and (5) M is not a non-abelian p-group with $|M:M'|<4q^2$, then the $k/q+1$ characters of G induced by those of M give rise, in a natural way, to k/q irreducible characters of G. This in a sense refines the "exceptional character" theory of R. Brauer and M. Suzuki [Suzuki, Amer. J. Math. **77** (1955), 657–691; MR **17**, 580]. Two important applications (Theorems 2, 3) are given, together with various corollaries. Theorem 2: Let G be a permutation group on $m+1$ letters in which no non-trivial permutation leaves three letters fixed (the subgroup leaving one letter fixed is then a Frobenius group N of order mq, where $|G|=(m+1)mq$). If G contains no normal subgroup of order $m+1$, then $m=p^e$ for some prime p and $|M:M'|<4q^2$, where M is a Sylow p-subgroup of G. (The natural conjecture that G contains a subgroup of index $\leqq 2$ isomorphic to LF $(2, p^e)$ has since been proved false, at least for $p=2$, by M. Suzuki [#11038].) Theorem 3: Let G be a group of even order. Assume that for any element u of order 2 contained in a Sylow 2-subgroup S_2, the centralizer of u is contained in the centralizer of S_2. Then at least one of the following holds: (a) S_2 is cyclic; (b) $S_2\lhd G$; (c) $G\cong D\times L$, where $|D|$ is odd and $L\cong$ LF $(2, 2^a)$ for some $a>1$. This generalizes a theorem of M. Suzuki, who assumed that the centralizers of the elements of order 2 are abelian [Trans. Amer. Math. Soc. **92** (1959), 191–204; MR **21** #7252]. No proofs are given, but these have since been published in Illinois J. Math. **4** (1960), 170–186; Amer. J. Math. **82** (1960), 281–300 [MR **22** #4784, 4785].

G. E. Wall (Sydney)

11038:

Suzuki, Michio. A new type of simple groups of finite order. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 868–870.

The author has constructed explicitly an infinite series of simple groups $G(q)$ of orders $q^2(q-1)(q^2+1)$, $q=2^{2n+1}$, n an integer >0. Since no order is divisible by 3, these groups are indeed new. They were found as subgroups of the general linear group of dimension 4 over the Galois field with q elements. The group contains the matrix $B=e_{14}+e_{23}+e_{32}+e_{41}$ and all matrices of either of the forms

$$M(Z)=Z^\theta\oplus Z^{1-\theta}\oplus Z^{\theta-1}\oplus Z^{-\theta},$$

$$S(\alpha,\beta)=\begin{bmatrix} 1 & & & \\ \alpha^\theta, & 1, & & \\ \beta, & \alpha, & 1, & \\ \alpha^{2\theta+1}+\alpha^\theta\beta+\beta^{2\theta}, & \alpha^{\theta+1}+\beta, & \alpha^\theta, & 1 \end{bmatrix},$$

with $\alpha^\theta=(\alpha)^r$, $r=2^n$. $\{S(\alpha,\beta)\}$ is a group of order q^2.

The nonexistence of $G(q)$ had been previously conjectured by Zassenhaus and Feit. $G(q)$ is (can be) a doubly transitive group on q^2+1 letters in which every element $(\neq 1)$ displaces at least q^2-1 letters. Every centralizer $(\neq G(q))$ of a single element in $G(q)$ is nilpotent. The (apparently computational) proof that $G(q)$ is actually simple is not given.

J. L. Brenner (Palo Alto, Calif.)

11039:

Curzio, Mario. Alcuni criteri di finitezza per i gruppi a condizione massimale o minimale. Ricerche Mat. **9** (1960), 248–254.

The author makes the observation that the groups of the following classes turn out to be finite: complemented soluble groups with maximal or minimal condition; complemented groups with minimal condition in which the two complementary factors are always permutable; soluble self-dual groups with maximal or minimal condition; relatively complemented locally finite groups with minimal condition. The proofs are very simple.

K. A. Hirsch (Boulder, Colo.)

11040:

Černikov, S. N. Infinite locally finite groups with finite Sylow subgroups. Mat. Sb. (N.S.) **52 (94)** (1960), 647–652. (Russian)

Le travail de l'auteur est dédié aux groupes infinis localement finis, c'est-à-dire, aux groupes dont un sous-ensemble fini quelconque d'éléments engendre un groupe d'ordre fini. Sous des hypothèses complémentaires (que nous irons préciser) l'auteur démontre qu'un groupe infini localement fini, G, ayant des sous-groupes de Sylow finis: (a) possède des sous-groupes véritables avec index fini (Théorème 2); (b) possède des sous-groupes abéliens infinis (Théorème 3). Les hypothèses complémentaires sont les suivantes: dans le cas (a), G possède un système normal, M, de degré fini par égard à un nombre premier p (cela veut dire que M possède au moins un facteur contenant des éléments dont la période est une puissance de p, mais qu'il ne possède pas de facteurs infinis jouissant de cette propriété; pour la définition de "système normal" nous renvoyons au livre de A. G. Kurosch, *Gruppentheorie*, Akademie-Verlag, Berlin [MR **15**, 681], p. 117); dans le cas (b), G possède un système normal à facteurs finis; G possède aussi des sous-groupes de Sylow finis pour un ensemble infini de nombre premiers. L'auteur souligne qu'il y a un problème ouvert, à savoir: les énoncés (a) et (b) sont-ils valables aussi dans le cas d'un groupe infini localement fini quelconque?

L. Lombardo-Radice (Rome)

11041:

Morris, Ifor. Modular orthogonal groups in four

dimensions. Proc. Cambridge Philos. Soc. **57** (1961), 239–246.

The author considers the 4-dimensional orthogonal groups (and the extended groups of orthogonal similitudes) over the real field and the Galois fields of odd characteristic. He derives explicitly the isomorphisms of these groups with 2-dimensional classical groups and their direct products. *G. E. Wall* (Sydney)

11042:

Reiner, Irving. Subgroups of the unimodular group. Proc. Amer. Math. Soc. **12** (1961), 173–174.

The congruence group $\Gamma(m)$ (for the modulus m) is the set of all 2×2 properly unimodular matrices which are congruent to the identity modulo m; $\Gamma(m)$ is a normal subgroup of $\Gamma\equiv\Gamma(1)$. It is known that Γ contains normal subgroups which do not contain any congruence group. The author proves, in particular, that the least normal subgroup of Γ which contains $\begin{bmatrix}1 & 6\\ 0 & 1\end{bmatrix}$ contains no congruence group; the same conclusion is valid if 6 is replaced by any integer which is not a prime power. The cases 2, 3, 4, 5 are special, as shown by the reviewer [Ann. of Math. (2) **71** (1960), 210–223; MR **22** #1622]. *J. L. Brenner* (Pala Alto, Calif.)

11043:

Nasu, Toshio. A note on homeomorphisms and fundamental groups. J. Sci. Hiroshima Univ. Ser. A **22**, 175–186 (1958).

Let $A=(a_{ij})$ be an $(n\times n)$-matrix with determinant one and integral entries. This paper is concerned with the totally discontinuous group Γ on the $(n+1)$-dimensional euclidean space R^{n+1} generated by the transformations S_i: $(x_1, \cdots, x_i, \cdots, x_{n+1})\to(x_1, \cdots, x_i+1, \cdots, x_{n+1})$, S_{n+1}: $(x_1, \cdots, x_n, x_{n+1})\to(x_1', \cdots, x_n', x_{n+1}+1)$ $(i=1, 2, \cdots, n)$ where $x_1, x_2, \cdots, x_{n+1}$ denote the coordinates of R^{n+1} and $x_i'=\sum a_{ij}x_j$. The author studies the following two problems: (1) For two such matrices A and A', what is the necessary and sufficient condition in order that the corresponding groups Γ and Γ' be isomorphic in the sense of abstract groups? (2) What is the condition in order that the two groups Γ and Γ' be linearly equivalent, i.e., that there exist a linear transformation U of R^{n+1} such that $U\Gamma U^{-1}=\Gamma'$? *H. C. Wang* (Evanston, Ill.)

11044:

Coxeter, H. S. M. Twelve points in PG(5, 3) with 95040 self-transformations. Proc. Roy. Soc. London. Ser. A **247** (1958), 279–293.

The group of order 1440, consisting of the automorphisms of the symmetric group S_6, is described in detail; it is the automorphism group for Tutte's eight-cage. The chords of a certain non-ruled quadric Q in PG(3, 3) are then identified with the edges of the eight-cage, and the ten points of Q are in turn derived from a set of six points in PG(4, 3); in fact, this derivation can be carried out in two ways, thus giving two four-spaces. The two sets of six points give a configuration A of twelve points in PG(5, 3); A can be divided into two sets of six suitable points in 66 ways and the whole complex of 132 sets of six points forms a geometric representation of the Steiner system $\mathfrak{S}(5, 6, 12)$. Numerous other geometric relations arise in the course of the discussion. *R. G. Stanton* (Waterloo, Ont.)

11045:

Todd, J. A. On representations of the Mathieu groups as collineation groups. J. London Math. Soc. **34** (1959), 406–416.

This paper establishes a geometric relation between M_{12} and M_{24}, namely, the automorphism groups of the Steiner systems $\mathfrak{S}(5, 6, 12)$ and $\mathfrak{S}(5, 8, 24)$ respectively. Coxeter [#11044] established a representation of M_{12} by a collineation group in PG(5, 3). The author discusses this further and shows that it leads to a representation of M_{24} with M_{12} as a sub-group. He also discusses M_{24} as a collineation group in PG(11, 2). *T. G. Room* (Sydney)

11046:

Edge, W. L. A setting for the group of the bitangents. Proc. London Math. Soc. (3) **10** (1960), 583–603.

To the quadratic form $\sum_{i<j} x_ix_j$ in seven variables over the field **F** of two elements there corresponds a quadric Q in [6], the six-dimensional projective space over **F**. The group Γ of collineations of [6] which leave Q invariant is isomorphic to the simple group of the twenty-eight double tangents to a plane curve of degree four; the order of Γ is 1 451 520. The quadric Q consists of 63 points. Of the remaining 64 points one, the kernel of Q, is contained in each of the 63 tangent [5]'s (hyperplanes) to Q. The author gives a complete description, too complex to be repeated here, of the various configurations formed by the intersections of Q with the subspaces of any dimension in [6]. This description yields geometric interpretations for various properties of Γ of which we mention only one simple example: For each tangent [5] to Q there is a unique (necessarily involutory) elation with this [5] as axis in Γ. The center of this elation is not on Q. Two distinct such elations commute if and only if the line joining their centers is a tangent to Q. Other interpretations are given for all elements of orders 2 and 3 in Γ; also the orders of the normalizers of various elements in Γ are determined. *P. Dembowski* (Frankfurt a.M.)

11047:

Klingenberg, Wilhelm. Linear groups over local rings. Bull. Amer. Math. Soc. **66** (1960), 294–296.

A local ring L is a commutative ring with unit which contains a maximal ideal. Without giving details, the author classifies invariant subgroups of the general linear group over L. The classification is similar to that obtained by the reviewer for a special case [Ann. of Math. (2) **39** (1938), 472–493; **45** (1944), 100–109; MR **5**, 228]. There are interesting rings not covered by these results, e.g., Euclidean rings. *J. L. Brenner* (Palo Alto, Calif.)

11048:

Auslander, Louis. Discrete solvable matrix groups. Proc. Amer. Math. Soc. **11** (1960), 687–688.

There exists an integer-valued function $g(n)$ of n, such that any solvable linear group G of degree n over the real field (G discrete in the usual topology) can be generated by less than $g(n)$ elements. The proof makes use of

topological considerations and results of H. C. Wang [Ann. of Math. (2) **64** (1956), 1–19; MR **17**, 1224]. The result also is implicitly contained in a paper of the reviewer [Abh. Math. Sem. Hansischen Univ. **12** (1938), 289–312]. *H. Zassenhaus* (Notre Dame, Ind.)

11049:

Zakon, Elias. Generalized archimedean groups. Trans. Amer. Math. Soc. **99** (1961), 21–40.

The concept of a regularly ordered abelian group was introduced by Robinson and Zakon [same Trans. **96** (1960), 222–236; MR **22** #5673] who made a metamathematical study of these groups. In this paper the author proves the theorems that were stated without proof in the above study, and he further develops the theory of these groups by algebraic methods. An o-group G is regularly ordered if for every infinite convex subset S of G and every positive integer n, there exists an element g in G such that $ng \in S$. Every archimedean o-group is regularly ordered and so is every divisible abelian o-group. For an abelian o-group A and a positive integer n let $[n]A$ be the order of the group A/nA. If A is discretely ordered, then A is regularly ordered if and only if $[n]A=n$ for all $n>0$. Let $p_1, p_2, \cdots$ be the ascending sequence of all primes and let $m_1, m_2, \cdots$ be an arbitrary sequence with each m_i either a non-negative integer or ∞. A densely ordered archimedean o-group M is constructed such that $[p_i]M=p_i^{m_i}$ for $i=1, 2, \cdots$. Let A and B be two disjoint countable regularly and densely ordered abelian groups with $[p]A=[p]B$ for all primes p. Then A and B can be embedded in a regularly and densely ordered abelian group M in such a way that $[p]M=[p]A$ for all primes p and that A and B are basic and pure subgroups of M.

The middle third of the paper is concerned with solving linear equations, inequalities, congruences and incongruences in abelian ordered groups, and, in particular, in regularly ordered abelian groups.

From a metamathematical standpoint there is good reason to refer to regularly ordered groups as generalized archimedean groups (see the review cited above), but it is apparent from this paper that there is a close connection between regularly ordered groups and divisible groups. The following two theorems illustrate this connection and can be used to simplify some of the proofs in this paper. An o-group G, with all convex subgroups normal, is regularly ordered if and only if G/C is divisible for all non-zero convex subgroups C of G. If H is an o-group with a convex subgroup Q that covers 0, then H is regular if and only if H/Q is divisible. In particular, a regularly and discretely ordered abelian group is an extension of an infinite cyclic group by a rational vector space, and a regularly ordered abelian group is divisible if and only if it contains a non-zero divisible convex subgroup. Thus a regularly ordered group is "almost" divisible, and hence if abelian it is close to being a rational vector space. This seems, to the reviewer, to be the basic fact that underlies the theory in this paper.

P. F. Conrad (New Orleans, La.)

11050:

Chehata, C. G. On a theorem on ordered groups. Proc. Glasgow Math. Assoc. **4**, 16–21 (1958).

The author gives a simple proof of a classic theorem on archimedean ordered groups, namely: Each archimedean ordered group is commutative. After defining in the usual manner the infinitely small elements, with respect to a given element, of a group G, the author shows that the theorem is a corollary of the following statement: In each abelian group the absolute value of the commutator $[a, b]=a^{-1}b^{-1}ab$ is infinitely small with respect to $\max(|a|, |b|)$. From it the assertion follows at once.

G. Gemignani (Pisa)

11051:

Lehner, Joseph. Representations of a class of infinite groups. Michigan Math. J. **7** (1960), 233–236.

The author proves that the free product of countably many cyclic groups of arbitrary orders q_j ($j=1, 2, \cdots$) has a faithful representation by 2-by-2 unimodular matrices whose entries are integers in the field generated by the numbers $2\cos \pi/q_j$ corresponding to finite q_j. After assigning a suitable matrix as a generator for each cyclic group, the author proves freedom geometrically by interpreting the matrices as bilinear transformations of the complex plane. *R. Steinberg* (Los Angeles, Calif.)

11052:

Fong, P. On the characters of p-solvable groups. Trans. Amer. Math. Soc. **98** (1961), 263–284.

In the general modular theory of finite groups each ordinary character χ_μ of a group G of order $p^a g_0$ can be decomposed into a sum of irreducible modular characters (mod p), φ_ρ,

$$\chi_\mu = \sum d_{\mu\rho}\varphi_\rho.$$

The rectangular matrix $D=[d_{st}]$ leads to $C=\tilde{D}D=[C_{st}]$, where the terms $C_{\rho\sigma}$ are the Cartan invariants of G. The matrix C is expressible as a direct sum of matrices each corresponding to a block of characters of G. Each block B of ordinary characters χ_μ has attached to it a p-subgroup of G of order p^d, which is called the defect group; and the block is said to be of defect d. The degree of χ_μ is divisible by p to exponent $a-d+\varepsilon_\mu$, where ε_μ is defined as the height of χ_μ.

The author, concerned especially with p-solvable groups, proves that if the defect group of a block B is abelian, then every character of B has height zero, and conversely if every character in the block which contains the unit character is of height zero then the defect group is abelian. This gives, incidentally, necessary and sufficient conditions for the Sylow p-subgroups of G to be abelian.

For p-solvable groups he proves that the Cartan invariants of a block B of defect d satisfy $C_{\rho\sigma} \leq p^d$, and obtains various other assorted results for these groups.

D. E. Littlewood (Bangor)

11053:

Gabriel, J. R. On the construction of irreducible representations of the symmetric group. Proc. Cambridge Philos. Soc. **57** (1961), 330–340.

The author presents a practical method of calculating the irreducible representations of the symmetric group $\pi(N)$ on N symbols. The procedure is especially adapted to the use of fast computing machines. The work was developed in connection with certain problems of electron interactions, and for this purpose it suffices to obtain the irreducible representations that correspond to a partition of N into two parts, say $N=(N-k)+k$. Such a representation is briefly denoted by (N, k), and in this paper details

are given for this type of representation only. But the method can be generalized to arbitrary irreducible representations. The matrices of (N, k) are obtained in real orthogonal form. Since the transpositions $(M-1, M)$ $(2 \leq M \leq N)$ generate the group $\pi(N)$, it is sufficient to calculate the matrices $P(M-1, M)$ representing these transpositions. It is not possible to describe all the details of the construction, but some features are worth mentioning. The group $\pi(N)$ has a sequence of subgroups $\pi(N-1)$, $\pi(N-2)$, $\cdots$, where $\pi(N-r)$ is the subgroup leaving the last r symbols invariant. The basis of the representation space of (N, k) is chosen in such a way that matrices representing elements of $\pi(N-r)$ appear in fully reduced form. Furthermore, $P(M-1, M)$ commutes with all matrices of $\pi(M-2)$. Schur's Lemma, together with certain character relations, deducible from Frobenius's generating function, can then be invoked to complete the calculations. *W. Ledermann* (Manchester)

11054:

Ляпин, Е. С. [Lyapin, E. S.]. ★Полугруппы [Semigroups]. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. 592 pp. 17.40 r.

A semigroup is any nonvoid set $\mathfrak{A}$ with an associative binary operation, written here as ordinary multiplication. The volume under review is a survey of the algebraic theory of semigroups as of the year 1960 (topological semigroups are not considered except in the bibliography). As the author points out, semigroups were considered early in the development of the theory of groups [see for example F. Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, Springer, Berlin, 1926–27; Teil I, Kapitel 8], but were put aside because of the inadequacy of the then available algebraic techniques. The author clearly, and perhaps even a little defensively, states his belief in the importance of semigroups, maintaining that while group theory is the abstract form of the theory of one-to-one mappings of a set onto itself, semigroup theory is the abstract form of the theory of single-valued mappings of a set into itself. Analysis, algebra, geometry, and topology being rich in examples of the latter, their abstract theory deserves recognition. With this viewpoint the reviewer is in wholehearted agreement.

The table of contents is: Chapter I, the concept of a semigroup; Chapter II, divisibility of elements; Chapter III, multiplication of subsets; Chapter IV, ideals; Chapter V, semigroups with minimal ideals; Chapter VI, invertibility; Chapter VII, homomorphisms; Chapter VIII, decompositions of semigroups into unions of subsemigroups; Chapter IX, relations in semigroups; Chapter X, imbedding of semigroups.

The book does not, of course, contain every known fact about semigroups. It does, however, offer a very complete introduction to the subject, and will be widely used both as a text and a reference. *E. Hewitt* (Seattle, Wash.)

11055:

Šutov, È. G. Defining relations in finite semigroups of partial transformations. Dokl. Akad. Nauk SSSR **132** (1960), 1280–1282 (Russian); translated as Soviet Math. Dokl. 1, 784–786.

Let N be the set of integers $1, \cdots, n$ $(n \geq 4)$. The author derives a system of defining relations for the semigroup W of all partial transformations of N, and for the inverse subsemigroup V of all the one-to-one partial transformations of N. Let α be the transformation of N that maps 1 and 2 upon 1 and is the identity on the rest of N. Let β be the identity mapping on the subset $2, \cdots, n$ of N. Let c_i be the permutation $(1\ i)$ of N for $2 \leq i \leq n$. Then $c_2, \cdots, c_n, \beta$ is an irreducible set of generators for V, and this set together with α is an irreducible set of generators for W. The above-mentioned defining relations are given in terms of these sets of generators, and they are quite elegant. *P. F. Conrad* (New Orleans, La.)

11056:

McFadden, R.; Schneider, Hans. Completely simple and inverse semigroups. Proc. Cambridge Philos. Soc. **57** (1961), 234–236.

Let S be a simple semigroup. The authors obtain seven conditions each of which is equivalent to S being both completely simple and inverse. If, as pointed out in the paper reviewed below [#11057], S does not contain a zero, then each of these conditions is equivalent to S being a group. The four conditions that do not explicitly assume that S is completely simple are: (1) S contains at least one non-zero idempotent and the product of any two distinct idempotents is zero. (2) For each $0 \neq a \in S$ there exists a unique x in S such that $axa = a$. (3) For each nonzero $a \in S$ there exist unique elements e and e' such that $ea = a = ae'$. (4) Every non-zero principal left ideal and every non-zero principal right ideal contains just one non-zero idempotent. {A semigroup is inverse if and only if every principal right ideal and every principal left ideal contains a unique idempotent generator. Using this and the fact that S is completely simple if and only if it contains a primitive idempotent, and that S is inverse if and only if the idempotents commute, the proofs of these equivalences can be shortened.} *P. F. Conrad* (New Orleans, La.)

11057:

Burgess, D. C. J. Note on the preceding paper. Proc. Cambridge Philos. Soc. **57** (1961), 237–238.

The "preceding paper" is the one reviewed above [#11056]. The following theorem is proved: If S is a completely simple inverse semigroup with zero, then either S is the union of a group with zero together with zero semigroups, or for each x in S, either $x^2 = 0$ or there exist a and b in S with $a^2 = b^2 = 0$ such that $x = ab$. {Munn [same Proc. **53** (1957), 5–12; MR **18**, 489] showed that S is the same as a Brandt semigroup, the structure of which was given by Clifford [Amer. J. Math. **64** (1942), 327–342; MR **4**, 4]. Thus $S = G \times I \times I$, where G is a group with zero and I is a set, and $(a, i, j)(b, k, m) = (ab, i, m)$ if $j = k$ and 0 otherwise. The theorem of the author is an immediate consequence of this. For if I contains just one element, then clearly S is a group with zero. If I contains at least two elements and $x^2 \neq 0$, then x has the form (a, i, i), and for any $j \neq i$, $(a, i, i) = (a, i, j)(e, j, i)$ and $(a, i, j)^2 = (e, j, i)^2 = 0$.} *P. F. Conrad* (New Orleans, La.)

11058:

Halezov, E. A. Automorphisms of primitive quasigroups. Mat. Sb. (N.S.) **53 (95)** (1961), 319–342. (Russian)

Let A be a six-operation free quasigroup with a finite

number of generators with operations xy, yx and the usual inverse operations x/y, $x\backslash y$, y/x, $y\backslash x$. Denote by $a_1, a_2, \cdots, a_n$ one of the bases of A. The author gives the group of automorphisms of A by a complete system of relations. The proof is based on the observation that every automorphism of A is a composite of automorphisms α_k for which also $\alpha_k a_i = a_i$ $(i \neq k)$, $\alpha_k a_k = a_k f P_k$ hold, where f is one of the operations of A and P_k is a polynomial built up from the elements $a_1, \cdots, a_{k-1}, a_{k+1}, \cdots, a_n$ by iteration of the operations of A. Similar statements are established for loops. The problem was raised by A. I. Mal'cev, and for monogenic loops it was solved by T. Evans [Proc. Cambridge Philos. Soc. **49** (1953), 579–589; MR **15**, 283].

M. Hosszu (Miskolc)

11059:

Belousov, V. D. The structure of distributive quasigroups. Mat. Sb. (N.S.) **50** **(92)** (1960), 267–298. (Russian)

Let $M(A)$ denote a quasigroup on a set M with binary operation $A(a, b) = ab = R_b a = L_a b$. It is by definition distributive if the laws $a \cdot bc = ab \cdot ac$, $bc \cdot a = ba \cdot ca$ are obeyed, and is a Moufang quasigroup if $a(b \cdot cb) = (ab \cdot c)b$ always. If k is any fixed element of M, the formula $B(a, b) = A(R_k^{-1}a, L_k^{-1}b)$ defines a loop $M(B)$ isotopic to $M(A)$, with unit element k. The author's fundamental theorem is that if $M(A)$ is distributive then $M(B)$ is a CML (commutative Moufang loop). The six associated quasigroups derived from $M(A)$ by applying the same permutation to a, b, c in every relation $ab = c$ are also distributive and have, relative to the same k, the same CML; relative to different elements k, l the CML's are isomorphic. The relation between $M(A)$ and $M(B)$ is preserved in subquasigroups, normal subquasigroups and quotient quasigroups. A condition is given (Lemma 8) that a given CML may correspond to some distributive quasigroup; and it is deduced, for example, that this is the case if in the CML the mapping $a \rightarrow a^2$ is a permutation, or if the CML is finite and of odd order. Many properties of distributive quasigroups are obtained by proving the corresponding properties of the CML; some examples are quoted below.

Let k again be an element of a quasigroup $M(A)$. The set of all permutations ϕ of M such that $(\phi a)b = a(\phi^* b)$ for a suitable permutation ϕ^* of M (depending on ϕ) and for all $a, b \in M$ is a group C_A. Then as ϕ runs through C_A, ϕk runs through a set denoted by $C_A(k)$ and called the (middle) associator of $M(A)$. A transitive quasigroup [cf. Belousov, Ukrain. Mat. Ž. **10** (1958), no. 1, 13–22; MR **20** #2390] is one for which $C_A(k) = M$ (all k). If $M(A)$ is distributive, then $C_A(k)$ is a transitive normal subquasigroup, and the quotient quasigroup $M(A)/C_A(k)$ is a distributive totally symmetric quasigroup. $C_A(k)$ then coincides as a set with $C_B(k)$, where $M(B)$ is the CML defined above; it is a group relative to the operation B, and is in fact the middle associator subgroup of $M(B)$ in the usual sense, i.e. (writing $B(a, b) = ab$), the set of elements x for which $ax \cdot b = a \cdot xb$ for all $a, b \in N$. If $M(A)$ is finite, distributive and anticommutative (i.e., $A(x, y) = A(y, x)$ only when $x = y$), then it is transitive. Any three elements of a distributive quasigroup generate a transitive distributive subquasigroup; so do any four elements a, b, c, d which satisfy the medial (or entropic) law $ab \cdot cd = ac \cdot bd$. The subquasigroup Q_k generated by all solutions x of $ab \cdot kd = ax \cdot bd$ where a, b, d run through a distributive quasigroup $M(A)$ is a distributive totally symmetric normal subquasigroup, and $M(A)/Q_k$ is transitive. This leads to another proof of the theorem given in the author's earlier paper [loc. cit.] that a distributive medial quasigroup is transitive. In that paper the question of whether there exist non-transitive distributive quasigroups was left open, but is now decided by the construction of two examples. Finally the author raises the question of whether there always exists a distributive quasigroup $M(A)$ with a given associator $C_A(k)$. In particular, he asks if $C_A(k)$ can consist of the single element k, and he shows that in this case the quasigroup, if it exists, must be totally symmetric.

I. M. H. Etherington (Edinburgh)

11060:

Berman, Gerald; Silverman, Robert J. Embedding of algebraic systems. Pacific J. Math. **10** (1960), 777–786.

The authors develop a representation theory for polyoids and polyrings. These are generalizations of groupoids and rings, having an arbitrary number of binary compositions $+_i$ operating on a set S, not necessarily on the whole of $S \times S$. In a (t, n)-polyring, for example, there are $t+n$ such compositions of which the last n are associative and each $+_j$ $(j = t+1, \cdots, t+n)$ is distributive with respect to $+_i$ $(i = 1, \cdots, j-1)$. In a polyoid, $n = 0$. The main theorem asserts that a (t, n)-polyring can be embedded isomorphically into a system of transformations on a set. Various known embedding theorems for systems such as groups, semigroups, rings, semirings, near-rings and associative neo-rings are contained in this result. Some further generalizations are indicated.

I. M. H. Etherington (Edinburgh)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 11255.

11061:

Mišík, Ladislav. Remarks on the U-axiom in topological groups. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **6** (1956), 78–84. (Czech. Russian summary)

A topological $\mathfrak{L}$-group is a set L with the property that it is an $\mathfrak{L}^*$-space and a group in which the group operation is continuous in both arguments and in which the inverse operation is continuous. A set $A \subset L$ is said to fulfil the v-axiom if, for every $B \subset A$,

$$\overline{\bar{B} \cap A} \cap A = \bar{B} \cap A,$$

where $\bar{A}$ denotes the (sequential) closure of A. Let $\delta(x, y)$ denote a metric defined in a topological $\mathfrak{L}$-group L, and let every sequence $\{x_n\}_{n=1}^{\infty}$ from L, for which there exists a sequence $\{y_n\}_{n=1}^{\infty}$ from L converging to an element $x \in L$ with $\lim \delta(x_n, y_n) = 0$, converge to x; then L is said to have property (d) for the metric $\delta(x, y)$.

It is shown that if L is a commutative $\mathfrak{L}$-group not fulfilling the v-axiom and if H is a dense subgroup of L, then, in order that H not fulfil the v-axiom, it is necessary and sufficient that for every $x \in L$ there exist a double sequence $\{x_{n,k}\}_{n,k=1}^{\infty}$ in H such that for every n the sequence $\{x_{n,k}\}$ converge to x but that x not be the limit of any diagonal sequence formed from $\{x_{n,k}\}$.

Suppose further that, for a topological $\mathfrak{L}$-group L, there exists a metric with the properties: (1) if $\lim x_n = x$,

$x_n \in L$, and if $x \in L$, then $\lim \delta(x_n, x) = 0$; (2) L has property (d) for $\delta(x, y)$. Then L does not fulfil the v-axiom.

The author concludes the paper with an example of an $\mathfrak{L}$-group not fulfilling the v-axiom but having a dense subgroup which does. *A. J. Lohwater* (Providence, R.I.)

11062:

Beck, Anatole. A note on semi-groups in a locally compact group. Proc. Amer. Math. Soc. **11** (1960), 992–993.

The following result is established: In a locally compact group, every semi-group of nonzero inner measure and finite outer measure is an open compact subgroup. This strengthens a result of Beck, Corson, and Simon [same Proc. **9** (1958), 648–652; MR **21** #697].

F. B. Wright (New Orleans, La.)

11063:

Tits, J. Sur une classe de groupes de Lie résolubles. Bull. Soc. Math. Belg. **11** (1959), 100–115.

The author starts from the following problem: Classify all connected Lie groups G having the property that every closed subgroup H is contained in a closed subgroup H_1 such that $\dim (H_1/H) = 1$. The solution: G has this property if and only if it has the following property: (S) There is a decreasing sequence $G = G_n \supset G_{n-1} \supset \cdots \supset G_0 = (1)$ of closed invariant subgroups such that the quotients G_{i+1}/G_i are all one-dimensional. Although the material dealt with here is strictly Lie group-theoretical, there are geometric applications arising from work of G. Valette [#11330 below].

This solution follows as a special case of the following property of connected Lie groups. Let $R^p(G)$ be the connected identity component of the intersection of all closed subgroups H such that there is no closed subgroup H_1 with $\dim (H_1/H) = 1$. There is a unique maximal closed connected invariant subgroup of G that has the property S; call it $R^s(G)$. Theorem: $R^p(G) = R^s(G)$.

The analogous results for Lie algebras and complex Lie groups are also considered. Lie's classification of all Lie groups acting effectively on the real line is used; in fact, the author gives a modern proof of Lie's results.

R. Hermann (Belmont, Mass.)

11064:

Mostow, G. D. Compact transformation groups of maximum rank. Bull. Soc. Math. Belg. **11** (1959), 3–8.

The author begins with a summary of some results on transformation groups and then proves the following: Let G be a compact Lie group acting almost effectively on a homological Euclidean space E, with dimension $E \leqq 2 \operatorname{rank} G + 1$. Then G has as its fixed-point set either a point or an open line. *D. Montgomery* (Princeton, N.J.)

11065:

Yang, Chung-Tao. p-adic transformation groups. Michigan Math. J. **7** (1960), 201–218.

Let G be a compact group acting effectively on a manifold X. It has long been conjectured that G is necessarily a Lie group. This question can be reduced to the question of whether or not compact p-adic groups can act on a manifold. The main result of this paper in this direction is the following. If G is a (compact) p-adic group acting on a manifold of dimension n, then the (homology) dimension of X/G is $n+2$. (That this apparently unlikely situation is not so unlikely as was first thought is demonstrated by an example of Raymond and Williams [Bull. Amer. Math. Soc. **66** (1960), 392–394] of a space (not a manifold) having this property.) Even if the space X is just a locally compact Hausdorff space of dimension n the author shows that $\dim X/G \leqq n+3$. *P. S. Mostert* (New Orleans, La.)

11066:

Oniščik, A. L. Complex hulls of compact homogeneous spaces. Dokl. Akad. Nauk SSSR **130** (1960), 726–729 (Russian); translated as Soviet Math. Dokl. **1**, 88–91.

Let K be a connected and compact Lie group. Its complex hull is defined to be the complex Lie group G containing K as a maximal compact subgroup such that the Lie algebra of G is the complexification of that of K. To each homogeneous space $X = K/L$ with L connected, the author associates a complex homogeneous space $Z = G/H$ where G, H are the complex hulls of K, L respectively. From the inclusion $K \subset G$, the space X is imbedded in Z as a submanifold. Among other results, the following are proved. (1) Let $o(X)$ be the algebra of all spherical functions over X (i.e., the complex-valued continuous functions f over X such that the totality of the transforms g^*f of f under the elements g of G spans a finite-dimensional linear space). Then there is a 1-1 correspondence between points of Z and homomorphisms of the algebra $o(X)$ into the field of complex numbers. (2) The manifold Z can be realized as a non-singular complex affine variety. *H. C. Wang* (Evanston, Ill.)

11067:

Dixmier, Jacques. Représentations intégrables du groupe de De Sitter. C. R. Acad. Sci. Paris **250** (1960), 4257–4259.

Let $\mathfrak{G}$ be the connected component of the identity of the orthogonal group of the quadratic form $x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2$. Let $\mathfrak{G}'$ be the (2-sheeted) universal covering group of $\mathfrak{G}$. Let $\mathfrak{R}$ be the subgroup of $\mathfrak{G}$ leaving x_1 fixed; similarly $\mathfrak{R}' \subset \mathfrak{G}'$. Then $\mathfrak{R}$ is a maximal compact subgroup of $\mathfrak{G}$, $\mathfrak{R}'$ of $\mathfrak{G}'$. The author studies the irreducible unitary representations ρ of $\mathfrak{G}'$, using the Lie algebra $\mathfrak{G}$ and the restriction of ρ to $\mathfrak{R}'$. This restriction decomposes into a direct sum $\sum \sigma_{k,k'}$ of (finite-dimensional) irreducible unitary representations $\sigma_{k,k'}$ of $\mathfrak{R}'$, which are naturally parametrized by two half-integers k, $k \geqq 0$. Let $\mathfrak{U}$ be the associative algebra generated by $\mathfrak{G}$. Then ρ is determined (up to unitary equivalence) by the set Γ of pairs k, k' occurring in $\rho(\mathfrak{R}') = \sum \sigma_{k,k'}$ and the complex number $\rho(\Omega)$, where Ω is the fundamental bilinear form considered as an element of $\mathfrak{U}$. The author gives a complete classification of the (equivalence classes) of representations ρ by determining all possibilities for Γ and $\rho(\Omega)$. The matrix coefficients of ρ satisfy differential equations. This is used to study their asymptotic behavior on $\mathfrak{G}'$ and to show that the matrix coefficients of certain ρ are in L^1 [resp. L^2]. Quite apart from the interest of these results to physicists, $\mathfrak{G}$ is the smallest-dimensional simple Lie group for which one thus knows the existence of square-integrable unitary irreducible representations without being able to write down representation spaces in a direct and explicit manner.

F. I. Mautner (Baltimore, Md.)

11068:

Harris, Bruno. Cohomology of Lie triple systems and Lie algebras with involution. Trans. Amer. Math. Soc. **98** (1961), 148–162.

A Lie triple system T is a subspace of a Lie algebra L closed under the ternary operation $[xyz]=[x,[y,z]]$ or, equivalently, it is the subspace of L consisting of those elements x such that $\sigma(x)=-x$, where σ is an involution of L. A T-module M is a vector space such that the vector-space direct sum $T\oplus M$ is itself a Lie triple system in such a way that (1) T is a subsystem, (2) $[xyz]$ is in M if any of x, y, z is in M, (3) $[xyz]=0$ if two of x, y, z are in M. A universal Lie algebra $L_u(T)$ and an $L_u(T)$-module M_s can be constructed in such a way that both are operated on by an involution σ and so that T and M consist of those elements of $L_u(T)$ and M_s which are mapped into their negatives by σ.

Now suppose L is a Lie algebra with involution σ and N is an L-σ-module. Then σ operates on $H^n(L,N)$ so that $H^n(L,N)=H_+{}^n(L,N)\oplus H_-{}^n(L,N)$ with both summands invariant under σ (details are omitted). The cohomology of the Lie triple system is defined by $H^n(T,M)=H_+{}^n(L_u(T),M_s)$. The author investigates these groups for $n=0,1,2$. $H^0(T,M)=0$ for all T and M; $H^1(T,M)\approx$ derivations of T into M modulo inner derivations; $H^2(T,M)\approx$ factor sets of T into M modulo trivial factor sets. Turning to the case of finite-dimensional simple T and ground field of characteristic 0, one has the Whitehead lemmas: $H^1(T,M)=0=H^2(T,M)$, and Weyl's theorem: Every finite-dimensional module is semi-simple. The paper ends by showing that if, in addition, the ground field ϕ is algebraically closed, then $H^3(T,\phi)$ is 0 or not 0, according as $L_u(T)$ is simple or not.

G. Leger (Cleveland, Ohio)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

See also 11356.

11069:

Anderson, L. W.; Ward, L. E., Jr. One-dimensional topological semilattices. Illinois J. Math. **5** (1961), 182–186.

The authors contribute to the general problem of the acyclicity of compact connected semigroups with zero by proving that if one such is commutative, idempotent, one-dimensional and locally connected, then it is a tree (dendrite). Moreover, such semigroups admit an algebraically induced partial order as well as a topologically induced partial order (the cutpoint order), and these are shown to be identical, a result of considerable interest in view of current activity in relation-theory. After learning of their result the reviewer [Bull. Amer. Math. Soc. **67** (1961) 123–124] was able to extend it somewhat.

A. D. Wallace (New Orleans, La.)

11070:

Hunter, R. P. On a conjecture of Koch. Proc. Amer. Math. Soc. **12** (1961), 138–139.

It is shown that the unit of a clan (which is not a group) is not a weak cutpoint. The one-dimensional case had been treated previously by R. J. Koch [Duke Math. J. **24** (1957), 611–615; MR **19**, 1064], who established the equivalence of the result stated above with several others.

A. D. Wallace (New Orleans, La.)

11071:

Lester, Anne. On the structure of semigroups with identity on a noncompact manifold. Michigan Math. J. **8** (1961), 11–19.

In what follows, a thread is a topological semigroup whose space is homeomorphic to an open interval, and an M-thread is a topological semigroup with identity whose space is homeomorphic to a half-open interval, and such that the endpoint acts as a zero. The following theorem is established.

Let M be an n-dimensional non-compact manifold $(n\geqq 2)$ which is a topological semigroup with identity 1, and not a group. Suppose there exists a compact connected $(n-1)$-dimensional group G containing 1 and contained in M. Then there is either a thread with identity or an M-thread, T, such that $M=TG$ and $tg=gt$ for $t\in T$, $g\in G$.

The proof is divided into twenty-two lemmas and uses many of the techniques of Mostert and Shields [Ann. of Math. (2) **65** (1957), 117–143; MR **18**, 809].

R. J. Koch (Baton Rouge, La.)

11072:

Matsushima, Yatarō. On the kernel of a topological semigroup with cut points. Proc. Amer. Math. Soc. **12** (1961), 20–23.

Let S be a connected topological lattice satisfying the modular law. For fixed elements a and b in S, define $B(a,b)$, the B-cover of a and b, by

$$B(a,b)=\{s|(s\vee a)\wedge(s\vee b)=(s\wedge a)\vee(s\wedge b)=s\}.$$

The author has previously investigated B-covers in a lattice and, in the present paper, investigates the structure of the kernel of a semigroup derived from a compact connected topological lattice. If multiplication is defined in S by $x\circ y=(a\vee x)\wedge(b\vee y)$, then it is shown that $(S,\circ)$ is a semigroup with $B(a,b)$ as its minimal closed ideal. Using results of L. W. Anderson concerning cut-points of a connected topological lattice, the author obtains a structure theorem in terms of lattice diagrams for the kernel in the case S has a point that cuts $B(a,b)$. This theorem is a special case of a theorem due to W. M. Faucett [same Proc. **6** (1955), 748–756; MR **17**, 173; Theorem 1.3]. The kernel is also described, though not completely, in the case that no-cut point exists.

Theorem 3 gives a necessary and sufficient condition for a two-sided ideal C of $(S,\circ)$ containing $B(a,b)$ to be prime in the case that there exists $z\in S$ such that $S\setminus z=C\cup D$, $C\neq\square\neq D$, $C^*\cap D=\square=C\cap D^*$. That S is connected should be assumed in the hypothesis of this theorem, for otherwise there are counter-examples.

Anne L. Hudson (New Orleans, La.)

FUNCTIONS OF REAL VARIABLES

See also 11311, 11267.

11073:

Phillips, E. G. ★A course of analysis. Cambridge University Press, New York, 1960. viii+361 pp. \$2.95; 15s.

A reprinting of the second edition which first appeared in 1939.

11074:

Dieudonné, J. ★Foundations of modern analysis. Pure and Applied Mathematics, Vol. X. Academic Press, New York-London, 1960. xiv+361 pp. $8.50.

This text, intended for first-year graduate or advanced undergraduate students, is supposed "to provide the necessary elementary background for all branches of modern mathematics involving 'analysis'". The treatment presupposes a knowledge of linear algebra and elementary classical calculus and a modest ability in epsilontics.

The most remarkable feature of the text is the consistently geometrical formulation of the results. For example, the differential calculus is developed in terms of linear approximation to functions on an open subset of a Banach space to a Banach space. Yet it would be completely false to assert that the book contained a study of Banach spaces—no non-trivial proposition on such spaces is proved. The subject of study is indeed elementary analysis, and the theorems are theorems of analysis stated in geometrical terms. This geometrization is rather like the geometrization of linear algebra which occurred some years ago, and, as in the linear algebra case, there are enormous conceptual and technical advantages. A good deal is accomplished in the 350 pages of the text. The mathematical organization is superb, the presentation lucid, there are a large number of very good problems, and there are excellent expository introductions to each chapter (couched in the author's customary diffident style). In brief, it is a beautiful text. What follows is a more detailed description of the contents.

Chapter I briskly introduces set-theoretic notation—set, products, mappings, unions and intersections, denumerability. Chapter II, equally brief, lists axioms for the real numbers (ordered Archimedean field with the nested interval condition) and elementary consequences thereof. Chapter III is devoted to the elementary topology of metric spaces: open and closed sets, cluster points, subspaces, continuity, limits, completeness, compactness, and connectedness. Chapter IV, concerning the real line again, deals with continuity of algebraic operations, logarithms and exponentials (a neat, intuititively satisfactory treatment as monotonic $(\cdot, +)$ isomorphisms and their inverses), and the Tietze-Urysohn extension theorem. Chapter V deals with elementary properties of normed spaces and Banach spaces: series (the author's definition of "series" here constitutes unnecessary cruelty to the reader), subspaces, finite products, multilinear mappings, spaces of multilinear mappings, linear forms, finite-dimensional spaces, and separability. Chapter VI, on Hilbert space, gives the minimal facts on hermitian forms, existence of orthogonal projections, and orthonormality. Chapter VII is on spaces of continuous functions. It establishes completeness properties, the Stone-Weierstrass theorem, and the Ascoli theorem. A special class of functions, called "regulated" (left- and right-hand limits exist at every point), is defined in preparation for the author's later omission of integration theory. Chapter VII, on differential calculus, defines the total derivative at a point as the approximating linear function. After establishing the formal rules of differentiation, the mean-value theorem (an inequality), integration (i.e., anti-differentiation), higher derivatives, and Taylor's formula are treated. In spite of the geometrical setting of the discussion, this is "hard" analysis, and a look at the problem lists will bring back memories of the Goursat texts. Chapter VIII concerns analytic functions. The initial sections apply to "Banach space valued" functions of n real or complex variables. (The author's better nature reasserts itself, and "series" and "sum of series" are used on p. 193 and thereafter in the rational way.) The exponential function is defined in terms of the power series for the real exponential, line integration is defined for roads (i.e., paths which are primitives of regulated functions), and the Cauchy theorem (homotopy form) is established for functions of one complex variable with values in a Banach space. The classical gambit (differentiability, singularities and zeros, residues, equicontinuity) follows. This chapter has a noteworthy appendix, based on Eilenberg's thesis, in which the Jordan curve theorem (with indices of the loop computed) and some separation theorems are established. Chapter X, local existence theorems, is devoted to the implicit-function theorem, the "rank" theorem, existence and continuity properties for solutions of ordinary differential equations, and the Frobenious theorem on total differential equations. The book ends with a flourish: Chapter XI concerns the Riesz theory of compact operators, Fredholm integral equations, and the Sturm-Liouville problem.

Finally, it is emphasized again that this is a text on elementary analysis. Thus, for example, there is no mention of the Lebesgue integral whose continuity and completeness properties make feasible the standard methods of Banach-space duality. The book is perhaps best characterized as a modern synthesis of basic classical analysis. *J. L. Kelley* (Berkeley, Calif.)

11075:

Leathem, J. G. ★Volume and surface integrals used in physics. Reprinting of Cambridge Tracts in Mathematics and Mathematical Physics, No. 1. Hafner Publishing Co., New York, 1960. vi+73 pp. $3.00.

A reprinting of the 1905 work [Cambridge Univ. Press, London].

11076:

Ostrowski, A. ★Vorlesungen über Differential- und Integralrechnung. Bd. 2: Differentialrechnung auf dem Gebiete mehrerer Variablen. 2te, neubearbeitete Aufl. Mathematische Reihe, Bd. 5. Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften. Birkhäuser Verlag, Basel-Stuttgart, 1961. 382 pp. Fr./DM 38.50.

The present revised edition of Vol. 2 is intended to follow the revised edition of Vol. 1 [1960; MR **22** #5701]. The exercises of the first edition have been removed entirely, as was the case in the revision of Vol. 1. Certain of the material of the first edition of Vol. 2 [1951; MR **13**, 540] concerning infinite series and the theory of curves is now in the revised edition of Vol. 1. Some new features of the revised edition of Vol. 2 follow. The notion of a matrix is introduced and its elementary properties are developed (§ 12). In the same section Jacobian matrices of a point transformation are considered and a sufficient condition for the univalence of a point transformation is treated. The study of the inversion of point transformations and the implicit-function theorem follow the approach of G. Kowalewski rather than the classical approach of Dini.

The Riemann-Stieltjes integral is introduced (§ 10) and is employed in the study of evolutes and for allied topics (§ 25). Bernoulli numbers and polynomials are treated in § 21 and the Euler-Maclaurin formula in § 22.

M. H. Heinz (Urbana, Ill.)

11077:

Mahavier, Wm. S. Rates of change and functional relations. Fund. Math. **48** (1959/60), 265–269.

If f and g are real functions on a common domain D, then f is said to be a function of g near x if x belongs to an open subset U of D such that $g(y)=g(y')$ implies $f(y)=f(y')$ for any y, y' in U; f is nowhere a function of g if f is not a function of g near any x in D [cf. Menger, Colloq. Math. **6** (1958), 41–47; MR **20** #6487]. Moreover [Menger, Fund. Math. **46** (1958), 89–102; MR **22** #85], for x in D, the rate of change $(df/dg)(x)$ exists if there is a number r with the following property: For every $\varepsilon>0$, there exists a $\delta>0$ such that for at least one number y we have $|y-x|<\delta$, $g(y)\neq g(x)$, while for every such y we have

$$\left|\frac{f(y)-f(x)}{g(y)-g(x)}-r\right| < \varepsilon.$$

There are examples due to Menger and to T. Engelhart which show that $(df/dg)(x)$ can exist for every x in D even though f is nowhere a function of g. In these examples (cited in the paper under review), either D contains no proper interval, or D contains a proper interval I but g is everywhere discontinuous on I.

The author elucidates these phenomena by proving the following theorem: If f and g are continuous real functions on a proper closed interval I, and if $(df/dg)(x)$ exists for every x in I, then f is a function of g near every y in a dense open subset of I. The proof of this theorem is based on several lemmas which are of interest in themselves; e.g., if f and g are continuous on I, and f is nowhere a function of g, then the plane curve $\{(g(x), f(x))|x \text{ in } I\}$ intersects some vertical line in infinitely many points.

A. Sklar (Chicago, Ill.)

11078:

Stancu, D. D. Sur quelques formules générales de quadrature du type Gauss-Christoffel. Mathematica (Cluj) **1** (**24**) (1959), no. 1, 167–182.

The author continues his study of quadrature formulas of Gauss-Christoffel type [Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **8** (1957), no. 1, 1–18; Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1** (**49**) (1957), 479–498; MR **20** #1875; **21** #3700]. He bases his work especially on a very general formula of the Gauss type due to T. Popoviciu [Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **6** (1955), 29–57; MR **19**, 64]. He generalizes this formula in the sense in which Christoffel generalized the Gauss formula, that is, he considers s multiple nodes fixed and tries to determine the other m nodes, of given order of odd multiplicity, in such a way that the quadrature formula thus obtained has the maximum degree of exactness.

E. Frank (Chicago, Ill.)

11079:

Smirnow, W. I. ★Lehrgang der höheren Mathematik. Teil III, 2. 3te, durchgesehene Aufl. Hochschulbücher für Mathematik, Bd. 4. VEB Deutscher Verlag der Wissenschaften, Berlin, 1961. xi+599 pp. DM 24.80.

For the first edition [1955], see MR **17**, 833.

11080:

Marcus, Solomon. Functions with the Darboux property and functions with connected graphs. Math. Ann. **141** (1960), 311–317.

There exists a (real-valued) function f (of a real variable) that satisfies the functional equation $f(x+y)=f(x)+f(y)$ and assumes each (real) value $2^{\aleph_0}$ times on every perfect set, whose graph G is connected, lacks the Baire property on every planar interval, and is such that for every measurable set E the set $G\cap E$ has inner measure zero and outer measure equal to the measure of E. There exists a function that assumes each value $2^{\aleph_0}$ times on every perfect set, is not Lebesque measurable, and lacks the Baire property, whose graph is totally disconnected. There exists a Lebesque measurable function that is not a Borel function and assumes each value $2^{\aleph_0}$ times in every interval, whose graph is totally disconnected.

F. Bagemihl (Detroit, Mich.)

11081:

Novikov, P. S.; Adyan, S. I. On a semicontinuous function. Moskov. Gos. Ped. Inst. Uč. Zap. **138** (1958), 3–10. (Russian)

A function defined on an interval is called essentially discontinuous if the interval cannot be decomposed into countably many sets on each of which it is continuous. Lusin raised the question whether there exist discontinuous B-measurable functions. The authors construct a semi-continuous, essentially discontinuous function.

A. Dvoretzky (Jerusalem)

11082:

Goffman, Casper; Waterman, Daniel. Approximately continuous transformations. Proc. Amer. Math. Soc. **12** (1961), 116–121.

The authors consider transformations $f(p)$ from Euclidean n-space E_n to a general metric space S. A measurable set $M\subset E_n$ is said to have metric density 1 at p if $m(M\cap Q)/m(Q)\to 1$ as $m(Q)\to 0$, where Q is an open n-cube containing p. The transformation f is said to be approximately continuous at a point p if, for every open set G containing $f(p)$, the set $f^{-1}(G)$ has metric density 1 at p. Denjoy [Bull. Soc. Math. France **43** (1915), 161–248] studied the properties of approximately continuous functions from E_1 to E_1 and Ridder [Fund. Math. **13** (1929), 201–209] extended the results to functions from E_n to E_1. In the present paper it is proved that, if f from E_n to S is approximately continuous, then (i) the set $f(E_n)$ is a separable subspace of S, and (ii) f is of Baire class 1.

A measurable set in E_n is called homogeneous if it has metric density 1 at each of its points. The authors consider a d-topology in E_n obtained by using the class of homogeneous sets as 'open' sets. This topology is Hausdorff but not Lindelöf. Among other results obtained, it is shown that every open connected subset of E_n is d-connected.

S. J. Taylor (Ithaca, N.Y.)

11083:

Iosifesku, Marius [Iosifescu, Marius]. Conditions that

the product of two derivatives be a derivative. Rev. Math. Pures Appl. 4 (1959), 641–649. (Russian)

Generalizing previous results of W. Wilkosz [Fund. Math. **2** (1921), 145–154] and G. Wolff [Nederl. Akad. Wetensch. Proc. **28** (1924), 282–285] the author gives conditions for the product fg of two derivatives f and g to be a derivative. The following cases are considered: f and g are bounded, f and g are finite and integrable, f is bounded and g is finite and integrable. If f is finite and $f \in L^2$ then f and f^2 are both derivatives if and only if $\lim_{h\to 0} h^{-1}\int_x^{x+h} |f(t)-f(x)|^2 dt = 0$ for every $x \in (a, b)$.

M. Cotlar (Buenos Aires)

11084:

Il'in, V. P. On a theorem of G. H. Hardy and J. E. Littlewood. Trudy Mat. Inst. Steklov. **53** (1959), 128–144. (Russian)

Let f be measurable on $[a, b]$ and of class Lip (α, p) where $0<\alpha\leq 1$, $p \geq 1$. That is,

$$\left\{\int_a^b |f(x+h)-f(x)|^p\,dx\right\}^{1/p} \leq Mh^\alpha$$

for some constant M and all $h>0$ (one may take $f(x)=0$ for x outside $[a, b]$). Hardy and Littlewood have shown that (i) if $\alpha p>1$ then f is equivalent to a function in Lip $(\alpha-1/p)$; (ii) if $\alpha p\leq 1$ then $f\in$ Lip $(\alpha-1/p+1/q, q)$ for every q between p and $p/(1-\alpha p)$ [Math. Z. **28** (1928), 612–634]. The author gives a new proof of these results based on the integral representation

$$f(x) = h^{-1}\int_0^h f(x+u)\,du - \int_0^h v^{-2}\,dv\int_0^v \{f(x+v)-f(x+u)\}\,du.$$

He gives explicit values for the constants which occur in the Lipschitz conditions. There are applications to sequences of functions satisfying integral Lipschitz conditions, and to other inequalities. The related inequality

$$|f(x)| \leq h^{-1/p}\|f\|_p + h^{1-1/p}\|f'\|_p$$

(with $p=2$) is used to obtain an upper bound for the eigenfunctions of certain Sturm-Liouville problems.

J. Korevaar (Madison, Wis.)

11085:

Ursell, H. D. Inequalities between sums of powers. Proc. London Math. Soc. (3) **9** (1959), 432–450.

Let $w_i>0$ $(i=1, \cdots, n)$, and put $S_\alpha = \sum_{i=1}^n w_i^\alpha$ for $\alpha>0$. The paper contains a penetrating study of the range of the sum S_α, given the values of similar sums $S_\beta, S_\gamma, \cdots$. When only one sum is given, Jensen's inequality furnishes the exact range of S_α. If, however, two sums are given then the convexity inequality, $S_\beta^{\alpha-\gamma} \leq S_\alpha^{\beta-\gamma} S_\gamma^{\alpha-\beta}$ for $\alpha>\beta>\gamma$, does not provide the exact range of the third sum (since equality occurs only when all w_i are equal, and this entails a relation between the two given sums). The question naturally gets more complicated as the values of more sums are prescribed (the value of n—which may be considered as S_0—may also be prescribed). The author proceeds by studying carefully the transformation $(w_1, \cdots, w_n) \to (S_{\alpha_1}, \cdots, S_{\alpha_m})$ using determinant inequalities and other tools. The results are too complicated to be reproduced here. Toward the end of the paper the author illustrates the general theory by determining the exact range of S_2 given S_1 and S_3.

A. Dvoretzky (Jerusalem)

11086:

Popoviciu, Tiberiu. Remarques sur la première et sur la seconde formule de la moyenne du calcul intégral. Mathematica (Cluj) **2 (25)** (1960), no. 1, 163–169.

11087:

Fast, H. Une remarque sur la propriété de Weierstrass. Colloq. Math. **7** (1959), 75–77.

For a real-valued function $f(x)$ of a real variable, what is here called the Weierstrass property is usually called the Darboux intermediate-value property: on every closed interval $[x', x'']$ in its domain the function $f(x)$ assumes every value intermediate between $f(x')$ and $f(x'')$. It is shown that to every real-valued function $F(x, y)$ defined on the whole (x, y)-plane there corresponds a real-valued function $u(x)$ defined on the whole real line such that for each real y_0 the function $F(x, y_0)+u(x)$ has the Weierstrass property. This yields at once the following theorem announced by A. Lindenbaum [Ann. Soc. Polon. Math. **6** (1927), 129–130] and proved in generalized form by W. Sierpiński [Matematiche, Catania **8** (1953), no. 2, 43–48; MR **16**, 229]: Every real-valued function on the real line is the sum of two functions having the Weierstrass property. The author's result also yields the following special case of a theorem of Sierpiński's [Matematiche, Catania **8** (1953), no. 2, 73–78; MR **16**, 230]: Every real-valued function on the real line is the pointwise limit of a sequence of functions having the Weierstrass property.

T. A. Botts (Charlottesville, Va.)

11088:

McLachlan, E. K. Extremal elements of a convex cone of subadditive functions. Proc. Amer. Math. Soc. **12** (1961), 77–83.

Let E be a set, let $\mathcal{S}$ be a class of subsets of E which contains E and the null set as members, and which is closed under set-union. In the vector space of real-valued functions on $\mathcal{S}$, consider the convex cone $\mathscr{C}$ of non-negative, monotone, subadditive functions. The author characterizes the extremal elements of the sub-cone $\mathscr{C}'$ of $\mathscr{C}$ consisting of those functions of $\mathscr{C}$ which have only a finite number of range values. Each extremal element of $\mathscr{C}'$ is an extremal element of $\mathscr{C}$. Whether or not the converse holds is an open question. Function classes like $\mathscr{C}$ arise in measure and capacity theory.

T. A. Botts (Charlottesville, Va.)

MEASURE AND INTEGRATION

See also 11080, 11081, 11082, 11234, 11251.

11089:

Maharam, Dorothy. Homogeneous extensions of positive linear operators. Trans. Amer. Math. Soc. **99** (1961), 62–82.

Let E_0 be a Boolean σ-algebra satisfying the countable chain condition, $F_0 = F(E_0)$ its function space, and φ_0 an F_0-integral on F_0, that is, a positive linear operator from F_0 to F_0 as previously defined by the author [same Trans. **75** (1953), 154–184; MR **14**, 1071]. It is shown that there exists an algebra E containing E_0 as a subalgebra, and an F-integral φ on $F=F(E)$ which is fully homogeneous (i.e., $x\in E$ and $0\leq g\leq \varphi(\chi_x)$ implies $\varphi(\chi_y)=g$ for

some $y \leq x$) and such that φ is an extension of φ_0 under the natural imbedding of F_0 in F. In the case that E_0 is a measure algebra with measure m_0, E can be so chosen that it is also a measure algebra, with measure m extending m_0. It is also shown that if A is any σ-subalgebra of a measure algebra (E, μ), and λ is a σ-finite positive measure on A, then λ can be extended to a σ-finite positive measure on E. *J. C. Oxtoby* (Bryn Mawr, Pa.)

11090:

Besicovitch, A. S. Tangential properties of sets and arcs of infinite linear measure. Bull. Amer. Math. Soc. **66** (1960), 353–359.

In this address, the author reviews the progress made in the present century in the study of tangential properties of sets in the plane. The existence of a tangent at almost all points of a rectifiable curve follows from the differentiability of a monotone function almost everywhere. Using linear measure, defined by Carathéodory, to generalize the concept of length, the author examined closely the geometry of (linearly) measurable plane sets of finite measure in a series of three papers [Math. Ann. **98** (1927), 422–464; **115** (1938), 296–329; **116** (1939), 349–357]. The main result obtained then about tangential properties come from the decomposition of any set of finite linear measure into the disjoint union of a regular set and an irregular set. Since regular sets are subsets of an enumerable union of rectifiable arcs they have nice tangential properties; further, the integral geometric measure of Favard (obtained by averaging the Lebesgue linear measure of the projection of the set on all directions) is zero for all irregular sets. Thus, in the sense of Favard measure, the tangent to a set of finite linear measure exists at almost all points. The author then shows how this main result can be extended to hold for any plane set of σ-finite linear measure. [Proc. Cambridge Philos. Soc. **52** (1956), 20–29; MR **17**, 595.]

S. J. Taylor (Ithaca, N.Y.)

11091:

Bzoch, R. C. Existence conditions for an integral of R. E. Lane. J. Indian Math. Soc. (N.S.) **23** (1959), 117–124 (1961).

The paper shows that a necessary and sufficient condition for the existence of a type of Stieltjes integral $\int_a^b f\,dg$ defined by R. E. Lane [Proc. Amer. Math. Soc. **6** (1955), 392–401; MR **16**, 911]—a generalisation of the mean Stieltjes integral—for f bounded and g of bounded variation on $[a, b]$, is that there exist a function $u(x)$ on $[a, b]$ and a subset G of $[a, b]$ containing a and b, such that the mean Stieltjes integral $\int_a^b u\,dg$ exists, $u(x)$ agrees with $f(x)$ on G, (g, f, H) with H the complement of G on $[a, b]$ is a singular graph and any point of discontinuity of g belonging to G is not an exceptional point for f and g on $[a, b]$. *T. H. Hildebrandt* (Ann Arbor, Mich.)

11092:

Darst, Richard B. A note on abstract integration. Trans. Amer. Math. Soc. **99** (1961), 292–297.

For an algebra S of subsets of a space X, a real-valued function f on X is defined to be continuous and to belong to $C(X, S)$ if, for every $\varepsilon > 0$, there exists a finite partition π of X into subsets in S such that the oscillation $\omega(f, E) < \varepsilon$ for each E in π. Then a necessary and sufficient condition that the Stieltjes integral $\int_X f\,dg$ by successive finite partitions relative to S exist for every finitely additive bounded function g on S is that f belong to $C(X, S)$. The algebra S determines also the classes of measurable functions $M(X, S)$, $LM(X, S)$, $RM(X, S)$ according as the sets $E[a < f < b]$, $E[a \leq f < b]$, $E[a < f \leq b]$ belong to S for all $-\infty \leq a < b \leq +\infty$, as well as their intersections with the space of bounded functions on X: $m(X, S)$, $Lm(X, S)$, $Rm(X, S)$. A necessary and sufficient condition that $m(X, s)$, $Lm(X, S)$ and $Rm(X, S)$ be equal, and equal to $C(X, S)$, is that S be a σ-algebra. If S is not a σ-algebra then $Lm(X, S) \neq Rm(X, S)$ and each is a proper subset of $C(X, S)$. $C(X, S)$ is the sum of $Lm(X, S)$ and $Rm(X, S)$ if and only if S has the property that, of two monotone increasing sequences of sets of S whose sums are disjoint, the sum of at least one of these sequences is a set of S. Essentially then, algebras of sets are natural for the Riemann-Stieltjes type of integrals, σ-algebras for the Lebesgue type. *T. H. Hildebrandt* (Ann Arbor, Mich.)

11093:

Nemitz, William C. On a decomposition theorem for measures in Euclidean n-space. Trans. Amer. Math. Soc. **98** (1961), 306–333.

This paper is a useful contribution to the study of geometrical properties of certain measurable subsets in Euclidean n-space. Besicovitch [Math. Ann. **98** (1927), 422–464] initiated this study when he defined regular and irregular sets in the plane, and showed that any linearly measurable plane set Q of finite linear measure can be decomposed into the union of a regular set Q_1 and an irregular set Q_2. In this context Q_1 is a subset of a countable union of rectifiable arcs, and Q_2 has the property that its orthogonal projection on almost all lines has Lebesgue measure 0. Federer [Trans. Amer. Math. Soc. **62** (1947), 114–192; MR **9**, 231] carried out a detailed study of subsets of E_n which he called k-rectifiable because they had tangential properties like those of 'nice' k-dimensional surfaces in E_n. Federer obtained a decomposition of measurable sets into three pieces, each with different tangential properties. Mickle [ibid. **92** (1959), 322–335; MR **22** #3792] simplified the Federer decomposition in the case $k = n-1$, showing that the Besicivitch decomposition has a direct generalisation to n-space.

The present paper modifies the methods of Mickle to show that the Besicovitch decomposition can be carried out for any positive integers k, n with $0 < k < n$. In doing this a measure μ_n^k is defined for all Borel sets in E_n with the following properties: (i) $F_n^k(B) \leq \mu_n^k(B) \leq H_n^k(B)$, where F_n^k is the integral geometric k-measure of Favard obtained by taking a suitable average of the Lebesgue measure of the projection onto all k-dimensional subspaces, and H_n^k is the k-dimensional Hausdorff measure; (ii) if B is k-rectifiable in the sense defined by Federer, then $F_n^k(B) = \mu_n^k(B) = H_n^k(B)$, while in general $\mu_n^k(B) < \infty \Rightarrow F_n^k(B) = \mu_n^k(B)$; (it remains an open question whether or not there exist any Borel sets for which $F_n^k(B) < \infty = \mu_n^k(B)$); (iii) if $k < m < n$, then $\mu_n^k(B) = \mu_m^k(B)$; (iv) μ_n^k is the smallest measure such that the Lebesgue k-measure of the projection into almost all k-subspaces of a set B is less than or equal to $\mu_n^k(B)$;

(v) for each B with $\mu_n{}^k(B) < \infty$, $B = B_1 \cup B_2$ where B_1 is countably k-rectifiable and $\mu_n{}^k(B_2) = 0$. The result (v) is the main object of the paper and it implies the direct generalisation of the Besicovitch decomposition theorem to the measure $H_n{}^k$. *S. J. Taylor* (Ithaca, N.Y.)

11094:

Billingsley, Patrick. Hausdorff dimension in probability theory. II. Illinois J. Math. **5** (1961), 291–298.

The author generalizes the results of a previous paper [same J. **4** (1960), 187–209], and relates the results to the Shannon-McMillan theorem of information theory, in the formgiven by L. Breiman [Ann. Math. Statist. **28** (1957), 809–811; MR **19**, 1148]. *I. J. Good* (Teddington)

11095:

Bognár, Mátyás. Bemerkungen zu einer Rede, die Friedrich Riesz bei Übernahme des Rektorats in Szeged gehalten hatte. Mat. Lapok **9** (1958), 232–259. (Hungarian. Russian and German summaries)

It is a question of the "principle" that a sequence of curves, each of length $\leq z$, cannot have a limit curve of length $> z$. Such a principle seems to have been used by F. Riesz [Mat. Fiz. Lapok **32** (1925), 112–124], in connection with the boundary of the union of an infinite set of discs of fixed radius. As the present author shows by a simple example, the principle can fail if the radius of the discs does not have a positive lower bound. With this restriction the principle can be justified in the following extended form: $\{\cdots, N_\alpha, \cdots\}$ is to be a system of plane sets, each component of an N_α being of diameter $\geq w$, for fixed $w > 0$. Denoting by H the boundary and by L^* the Carathéodory outer linear measure, assume that $L^*(H(\bigcup_{i=1}^l N_{\alpha_i})) \leq z$ for any finite collection N_{α_i}. Then for the whole system of sets one has $L^*(H(\bigcup_\alpha N_\alpha)) \leq z$. The proof takes place in seventeen stages, in the course of which some interesting subsidiary results are proved.

F. V. Atkinson (Toronto)

11096:

Farquhar, I. E. The present state of ergodic theory. Nature **190** (1961), 17–18.

11097:

Bartle, Robert G.; Joichi, James T. The preservaton of convergence of measurable functions under composition. Proc. Amer. Math. Soc. **12** (1961), 122–126.

Let ϕ be a scalar-valued Borel measurable function of a scalar variable. Let M denote a particular type of convergence for a sequence of measurable functions or of totally measurable functions. The function ϕ is said to preserve M-convergence if, given a sequence $\{f_n\}$ of measurable functions which is M-convergent to f, then the composition product $(\phi \circ f_n)$ is M-convergent to $(\phi \circ f)$. The authors characterize functions ϕ of this type for various types of convergence and in the first four theorems prove that (i) ϕ preserves M-convergence [M indicates almost everywhere convergence of sequences of measurable functions, or almost everywhere convergence or almost uniform convergence or convergence in measure, of sequences of totally measurable functions] if and only if ϕ is continuous and (ii) ϕ preserves M-convergence [M indicates uniform convergence, almost uniform convergence or convergence in measure of sequences of measurable functions or indicates uniform convergence of sequences of totally measurable functions] if and only if ϕ is uniformly continuous. They also examine the conditions under which ϕ preserves convergence in L_p, $1 \leq p < \infty$. *M. S. Ramanujan* (Ann Arbor, Mich.)

11098:

Kuroda, Tadashi. A criterion for a set to be of 1-dimensional measure zero. Japan. J. Math. **29** (1959), 48–51.

Let F be the complement of a compact set E. Consider a system of doubly connected regions $R_n{}^k$ in F, finitely many for each n. Every R_{n+1}^k lies in the bounded complement of some $R_n{}^h$, and for fixed n every point of E lies inside of some $R_n{}^k$, in the same sense. Let $\mu_n{}^k$ be the module of $R_n{}^k$, and set $\mu_n = \min_k \mu_n{}^k$. The author considers consequences of the assumption (1) $\sum \log \mu_n = \infty$, some of which are well known. If E is a linear set and each $R_n{}^k$ is bounded by two circles, symmetric to the line, he concludes from (1) that E has linear measure zero. For Cantor sets $E(p_1, p_2, \cdots)$ the condition is also necessary.

L. Ahlfors (Cambridge, Mass.)

FUNCTIONS OF COMPLEX VARIABLES

See also 11015, 11074, 11203, 11204, 11228, 11234, 11240, 11276, 11412.

11099:

Trohimčuk, Yu. Yu. Continuous mappings and analytic functions. Issledovaniya po sovremennym problemam teorii funkciĭ kompleksnogo peremennogo, pp. 7–29. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

A survey lecture, given at the 3rd All-Union Conference on Complex Variables in Moscow in 1957, concerning sufficient conditions that a continuous mapping be analytic in a domain. The report contains a systematic treatment with proofs of the author's earlier extensions [Uspehi Mat. Nauk **11** (1956), no. 5 (71), 215–222; Ukrain. Mat. Ž. **8** (1956), 177–190; Mat. Sb. (N.S.) **45** **(87)** (1958), 233–260; Dokl. Akad. Nauk SSSR **121** (1958), 430–431; MR **19**, 237, 537; **21** #119, 129] of the theory of Menshov and Bohr. *A. J. Lohwater* (Houston, Tex.)

11100:

Mel'nik, I. M. On topological methods in the theory of functions of a complex variable. Dokl. Akad. Nauk SSSR **131** (1960), 1015–1018 (Russian); translated as Soviet Math. Dokl. **1**, 372–375.

Results of Morse and others on critical points of harmonic functions are extended to the case of a function $u(x, y)$ which is the real part of a function $f(z)$ having singularities of the type of $z^k[p(z) \log^n z + q(z)]$, where k, n are integers and $p(z)$, $q(z)$ are analytic. Complications arise from the multiple-valued nature of $u(x, y)$. The results are presented without proof. *W. Kaplan* (Ann Arbor, Mich.)

11101:

Šabat, B. V. The modulus method in space. Dokl. Akad. Nauk SSSR **130** (1960), 1210–1213 (Russian); translated as Soviet Math. Dokl. **1**, 165–168.

The author extends the method of extremal length of Beurling and Ahlfors to three-dimensional space. The modulus of a family $\{C\}$ of rectifiable curves in a domain D is defined as $M\{C\}=\inf \int_D \rho^3 d\omega$, the "inf" being taken over all admissible metrics ρ (ρ = measurable, non-negative function in D with $\int_C \rho\, ds \geqq 1$ for all C). The modulus of a family $\{S\}$ of surfaces of finite area is likewise defined as $M\{S\}=\inf \int_D \rho^3 d\omega$ over all ρ with $\int_S \rho^2 d\sigma \geqq 1$ for every S. The moduli of some special families of curves and surfaces are computed. The well-known properties of moduli in the two-dimensional case (inequalities for moduli and their change under differentiable quasiconformal mapping, the unicity a.e. of an extremal metric) are established for the new notions. *K. Strebel* (Fribourg)

11102:

Šabat, B. V. Mappings effected by solutions of nonlinear systems of partial differential equations. Issledovaniya po sovremennym problemam teorii funkciĭ kompleksnogo peremennogo, pp. 451–461. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

If the system of equations

$$(1)\qquad F_i(x, y, u, v, u_x, u_y, v_x, v_y) = 0 \qquad (i = 1, 2)$$

is linear in (u_x, u_y, v_x, v_y), if the coefficients are bounded and measurable, and if the system is uniformly elliptic in the sense of Petrovsky, then it is known that any generalized solution in a region D, with first generalized derivatives square summable on every compact subset of D, is topologically equivalent to an analytic function. The author shows by example that this result fails when the system is non-linear. In the case that (1) is strongly elliptic in the sense of Lavrentiev, and the coefficients appearing in a certain transformation of (1) are bounded, the author proves that any solution $f(z)=u+iv$ with generalized second derivatives which are square summable on any compact subregion, is then topologically analytic. *R. Finn* (Stanford, Calif.)

11103:

Reischer-Haimovici, Corina. Fonctions en algèbres d'ordre fini. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I (N.S.) **5** (1959), 5–11. (Russian and Romanian summaries)

Let $\mathfrak{A}$ be a finite-dimensional, linear, associative algebra, with identity, over the complex field. The author makes an extension of functions of a complex variable to functions with domain and range in $\mathfrak{A}$, by use of the isomorphism of $\mathfrak{A}$ with the algebra of matrices of its right multiplications, and utilization of the classical theory of functions on complex matrices. {This approach and an alternative point of view were also discussed by the reviewer [see following review].} Elementary properties of the extension are proved. The author defines "derivative" and "integral" of an extension of $f(z)$ to $\mathfrak{A}$ as the respective extensions of $f'(z)$ and $\int f(z)dz$ to A. {These concepts seem rather barren in the absence of some concept of derivative and integral defined directly in the algebra $\mathfrak{A}$. A curious misstatement occurs to the effect that $a^{1/p}$, $a \in \mathfrak{A}$, is defined only if $a \neq 0$.} *R. F. Rinehart* (Cleveland, Ohio)

11104:

Rinehart, R. F. Elements of a theory of intrinsic functions on algebras. Duke Math. J. **27** (1960), 1–19.

Let A be a finite-dimensional linear associative algebra with identity over the complex numbers, and G the group of automorphisms and anti-automorphisms of A which leave the scalars fixed. A function $f(\xi)$ from A to A is called intrinsic if $f(\Omega\xi)=\Omega f(\xi)$ for all Ω in G. $f(\xi)$ is called primary if it is the extension to A, by means of a Cauchy integral formula or a power series, or more generally by a Hermite interpolation formula, of a scalar function of a scalar variable. It is shown that every primary function is intrinsic; and that when A is the set of quaternions over the complex numbers the notions coincide, and $f(x_0+p\mu) = u(x_0, p)+v(x_0, p)\mu$ if $f(x+iy)=u(x, y)+iv(x, y)$, where $p=(x_1^2+x_2^2+x_3^2)^{1/2}$, and μ is the unit vector parallel to the "imaginary part" $\sum_{i=1}^3 x_i j_i$ of the quaternion $x_0+\sum_{i=1}^3 x_i j_i$. Further, in this case $f(\xi)$ is Hausdorff analytic if the scalar function $f(x)$ is analytic in a domain in the complex x-plane and $f(\bar{x})=\overline{f(x)}$. The notions and results can be modified for algebras with real scalars by first extending the scalar field to the complex numbers. *J. B. Crabtree* (Hoboken, N.J.)

11105:

Pommerenke, Ch. On some metric properties of polynomials with real zeros. II. Michigan Math. J. **8** (1961), 49–54.

The paper is a continuation of the author's previous paper on the same subject [same J. **6** (1959), 377–380; MR **22** #759]. The author shows first: If F is a closed bounded set which is the union of circular discs orthogonal to the real axis, then $\Lambda \leqq \pi d \leqq 4\pi$ cap F, where Λ and d denote the sum of the perimeters and diameters, respectively, of the components of F. A special case of such a set F is the set E: $|f(z)|\leqq 1$, where $f(z)=\prod (z-x_\nu)$ is a polynomial with real zeros x_ν. In this case, cap $E=1$, hence $\Lambda \leqq 4\pi$. The author also obtains two theorems concerning upper estimates for the width of E: (a) in terms of $|f(0)|$ under the assumption that the zeros x_ν are symmetrical with respect to the origin; (b) without this assumption, in terms of $r=\operatorname{Max}|x_\nu|$.

{The following two minor changes should be noted by the reader: On page 49, line 13 from below, for $4d$ read πd, and on page 50, line 7 from below, for F_μ read K_μ.} *F. Herzog* (E. Lansing, Mich.)

11106:

Rahman, Q. I. Some inequalities for polynomials and related entire functions. Illinois J. Math. **5** (1961), 144–151.

This paper contains a number of inequalities for polynomials and for periodic entire functions of exponential type. *E. Frank* (Chicago, Ill.)

11107:

Wilf, Herbert S. Perron-Frobenius theory and the zeros of polynomials. Proc. Amer. Math. Soc. **12** (1961), 247–250.

The author proves that all the zeros of a polynomial $f(z)=a_0+a_1z+\cdots+a_nz^n$ lie in the circle

$$|z| \leq \max\{|a_{n-i}/a_n|(x_1/x_i)+(x_{i+1}/x_i)\} \qquad (i = 1, 2, \cdots, n),$$

where the x_i are arbitrarily chosen positive numbers for $i=1, 2, \cdots, n$ and $x_{n+1}=0$. This result includes some well-known bounds due to Cauchy, Kojima and Fujiwara, and follows at once from the well-known theorems of Perron and Frobenius on the characteristic roots of matrices. *M. Marden* (Milwaukee, Wis.)

11108:

Specht, Wilhelm. Zur Verteilung der Nullstellen komplexer Polynome. Math. Nachr. **21** (1960), 109–126.

Using known results about the eigenvalues of matrices the author proves: $\sum (|\zeta_\nu|-1)^2 \leqq \beta(\beta+4)$ [resp. γ^2], the summation extending over all n zeros of a polynomial $z^n + a_1 z^{n-1} + \cdots + 1$ whose coefficients satisfy $|a_\nu| \leqq \beta$ ($\nu = 1, \cdots, n$) [resp. $|a_1|^2 + \cdots + |a_{n-1}|^2 \leqq \gamma^2$]. Somewhat improved bounds are also obtained.

A. Dvoretzky (Jerusalem)

11109:

Cutteridge, O. P. D. Further theory of a certain continued fraction. Proc. Inst. Elec. Engrs. C **107** (1960), 234–237.

Author's summary: "The paper develops further theory of a certain type of continued fraction relevant to the problem of determining the character of the zeros of a polynomial. Two theorems provide tests for the number of positive zeros, real zeros and pairs of conjugate complex zeros of a real polynomial. Two numerical examples are included, one of which shows the application of the method to a problem in linear-network theory."

11110:

Zervos, Spiros. Sur la localisation des zéros des séries. C. R. Acad. Sci. Paris **249** (1959), 219–221.

In this note, the author generalizes some of the concepts in his earlier work [same C. R. **245** (1957), 394–396, 619–622; MR **19**, 948]. In part I, he generalizes the notion of a zero of a function to real- or complex-valued functions on a topological space. For these functions, he states a generalization of the Hurwitz theorem on the continuity of the zeros of a uniformly convergent sequence of functions. He also extends the theorems on the minorants of zeros of functions defined by certain series. In part II, he shows how minorant theorems about the zeros of polynomials can lead to such theorems for Taylor series. In part III, he cites several implications of the theorem of Hurwitz on the continuity of zeros of continuous functions.

G. Springer (Lawrence, Kans.)

11111:

Noble, M. E. A note on non-continuable power series. J. London Math. Soc. **35** (1960), 117–127.

Ein klassischer Satz von E. Fabry [Acta Math. **22** (1899), 65–87] besagt folgendes: Die Potenzreihe $\sum a_n z^n$ mit $\limsup |a_n|^{1/n}=1$ ist über $|z|=1$ hinaus nicht fortsetzbar, wenn es eine Folge $\{n_k\}$ natürlicher Zahlen gibt, sodaß $|a_n|^{1/n} \to 1$ strebt für $n=n_k\to\infty$ (a_{n_k} 'wesentliche' Koeffizienten), und im Intervall $|n-n_k| < \vartheta n_k$ ($\vartheta > 0$) höchstens $o(n_k)$ nicht verschwindende a_n existieren. Dieser Satz wurde verschiedentlich dahingehend verallgemeinert, daß die Forderung der Lückenlänge ϑn_k abgeschwächt und dafür weitere Bedingungen über $|a_n|$ aufgenommen wurden; vgl. vor allem Lösch [Math. Z. **32** (1930), 415–421] und Claus [ibid. **49** (1943), 161–191; MR **5**, 176]. Der Verf. beweist folgenden Satz von diesem Typ. Gegeben sei $\sum a_n z^n$ mit $\limsup |a_n|^{1/n}=1$, und $\phi(x)$ sei die kleinste konkave Majorante von $\log(|a_n|+2)$. Es seien n_k, N_k ganze Zahlen mit $n_k \leqq N_k < 2n_k \to \infty$, und von den a_n in $n_k \leqq n \leqq N_k$ sei folgendes gefordert. (1) $\phi(n_k) = o(N_k - n_k)$, Wachstumsbeschränkung für die a_n in $n_k \leqq n \leqq N_k$; (2) $\limsup |a_{n_k}|^{1/(N_k-n_k)} = 1$ oder $\limsup |a_{N_k}|^{1/(N_k-n_k)} = 1$, d.h., a_{n_k} oder a_{N_k} sind 'wesentlich'; (3) für große k und ein $\lambda < 1$ ist $|a_n| \leqq \lambda^{(N_k-n_k)}$ in $n_k < n < N_k$ außer für höchstens $o(N_k - n_k)$ Zahlen n. Dann ist $\sum a_n z^n$ über $|z|=1$ hinaus nicht fortsetzbar. Die bei Claus geforderte Bedingung, Lückenlänge/log $n \to \infty$, ist also überflüssig. Der Satz wird falsch, wenn in (2) $1/(N_k-n_k)$ durch $o(1)/(N_k-n_k)$ ersetzt wird, oder wenn in (1) O statt o steht. Der Beweis knüpft an Claus an und verwendet insbesondere Tschebischeff-Polynome, die zu einem Sektorgebiet gehören. *D. Gaier* (Pasadena, Calif.)

11112:

Gel'fer, S. A. On the maximum conformal radius of a fundamental domain of a group of fractional-linear transformations. Mat. Sb. (N.S.) **52 (94)** (1960), 629–640. (Russian)

The paper achieves essentially the same results as a previous paper with the same title published in Dokl. Akad. Nauk SSSR **126** (1959), 463–466 [MR **21** #5728].

B. A. Amirà (Jerusalem)

11113:

Batyrev, A. V. Conformal mapping of nearby regions. Issledovaniya po sovremennym problemam teorii funkciĭ kompleksnogo peremennogo, pp. 358–365. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

The author extends the method developed by M. A. Lavrent'ev [*Konformnye otobraženiya s priloženiyami k nekotorym voprosam mehaniki*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow-Leningrad, 1946] for the numerical determination of conformal maps onto the unit disk of nearly circular regions, star-shaped with respect to the origin. {This method, as described by the author, is very similar to Theodorsen's method, developed by S. Warschawski [Quart. Appl. Math. **3** (1945), 12–28; MR **6**, 207].} It leads, under suitable normalization of the mapping function, to the determination of the solution of the integral equation

$$\varphi(\theta) - \theta = -\frac{1}{2\pi}\int_0^{2\pi} \log r[\varphi(t)] \operatorname{ctg} \frac{t-\theta}{2}\, dt,$$

where the integral is taken in the sense of the Cauchy principal value and $r = r(\varphi)$ is the equation in polar coordinates of the boundary of the given region, θ and t being arguments of points on the unit circumference. A similar integral equation is obtained in the more general case, also considered by the author, when the region is no longer star-shaped. Both integral equations are then handled by the method of successive approximations starting with $\varphi(\theta) = \theta$ and a similar condition in the more general case. The author studies the principal value of the integral

$$\Phi(\theta) = \frac{1}{2\pi}\int_0^{2\pi} g(t) \operatorname{ctg} \frac{t-\theta}{2}\, dt$$

and estimates its order of smallness under various assumptions concerning the moduli of continuity of $g(t)$ and of $g'(t)$. This is then applied to estimate the error in the first approximation to the mapping function both in Lavrent'ev's case and in the more general case.
W. Seidel (Detroit, Mich.)

11114:

Lebedev, N. A. The area principle in the problem of non-overlapping regions. Dokl. Akad. Nauk SSSR **132** (1960), 758–761 (Russian); translated as Soviet Math. Dokl. **1**, 640–644.

The author obtains rather complicated sets of inequalities, including a generalization of the area principle, for a class of functions $w=f_k(z), f_k(0)=a_k$ $(k=0, 1, \cdots, n)$ where $a_k \in D_k$ a set of $n+1$ arbitrary pairwise disjoint simply connected domains. The function $f_k(z)$ maps $|z|<1$ conformally and univalently onto the region D_k. These inequalities are then used to achieve integral inequalities for functions univalent in $|z|<1$ which are of the form $f(z)=\sum_{n=1}^{\infty} a_n z^n$ and such that for any two points z_1 and z_2 in $|z|<1$ the product $f(z_1)\overline{f(z_2)} \neq -1$, or $f(z_1)f(z_2) \neq 1$.
W. C. Royster (Lexington, Ky.)

11115:

Mizumoto, Hisao. On conformal mapping of a multiply-connected domain onto a canonical covering surface. II. Kōdai Math. Sem. Rep. **12** (1960), 11–14.

Making use of the results of an earlier paper [same Rep. **10** (1958), 177–188; MR **21** #4234] on "covering surfaces of annular type" the author proves: Any plane domain B bounded by $N \geqq 1$ continua C_j can be mapped conformally onto a "covering surface G of circular type cut along concentric circular slits centered at the origin", i.e., a finite-sheeted, bounded covering surface of the w-plane with circles and circular slits centered in 0 as boundary components and which covers the origin. The zeros z_k with orders μ_k and the rotation numbers ν_k of the boundary components about the origin can be preassigned $(\sum_{j=1}^{N} \nu_j = \sum_{k=1}^{N'} \mu_k)$. The mapping is then determined up to a transformation $w \rightarrow aw$.
K. Strebel (Fribourg)

11116:

Mizumoto, Hisao. On conformal mapping of a Riemann surface onto a canonical covering surface. Kōdai Math. Sem. Rep. **12** (1960), 57–69.

The author considers finite Riemann surfaces R bounded by $r \geqq 1$ continua and of genus $g \geqq 0$ and the following three types of covering surfaces: n-sheeted disks F $(r \leqq n < \infty)$, covering surfaces of annular type cut along concentric circular slits G (all boundary components are circles or circular slits with centre 0 and 0 is not covered) and covering surfaces of circular type H [see #11115]. Using potential-theoretic and topological methods he proves the existence of conformal maps of R onto covering surfaces of class F and H. The mapping onto a covering surface of class G is not always possible if $g \geqq 1$. The radii of the boundary components of the surface H can be given in advance.
K. Strebel (Fribourg)

11117:

Dumkin, V. V.; Markov, M. N. The Riemann surfaces R_n corresponding to functions of the class $w=e^z+z^{-n}$ $(n=1, 2, 3, \cdots)$. Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, pp. 443–450. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

The Riemann surfaces referred to in the title are analyzed, first for $n=1$ and then for general n; the surfaces are decomposed into sheets bounded by lines $\operatorname{Im} w=$ const, and branch points are located.
W. Kaplan (Ann Arbor, Mich.)

11118:

Rodin, Yu. L. Boundary problems of the theory of analytic functions on Riemann surfaces of finite genus. Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, pp. 436–442. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

Let R be a Riemann surface of finite genus, let T_1 be a region on R bounded by C, formed of a finite number of smooth distinct Jordan curves; let $G(z)$ and $g(z)$ be continuous functions on C, with $G(z) \neq 0$. The author considers the Hilbert problem of finding functions $F_1(z)$, $F_2(z)$, analytic in T_1 and T_2, the interior of the complement of T_1, respectively, and such that on C, $F_1(z)=G(z)F_2(z)+g(z)$. Conditions are given for existence of solutions. {The reviewer found some details unclear.}
W. Kaplan (Ann Arbor, Mich.)

11119:

Kuroda, Tadashi. On some theorems of Sario. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2 (50)** (1958), 411–417.

Let F be an open Riemann surface and consider a canonical exhaustion $\{F_n\}$ with borders Γ_n. The modulus μ_{nk} of a component $\overline{R}_{nk}$ of $\overline{F}_n - F_{n-1}$ is defined by the harmonic function u on R_{nk} with $u=0$ on $\alpha=\Gamma_{n-1} \cap R_{nk}$, $u=\log \mu_{nk}$ on $\beta=\Gamma_n \cap R_{nk}$, $\int_\alpha du^*=2\pi$. If the product $\prod_1^\infty \mu_n$ of the minimal moduli $\mu_n=\min_k \mu_{nk}$ diverges, then F belongs to the class O_{AD} of Riemann surfaces not admitting non-degenerate analytic functions with a finite Dirichlet integral [the reviewer, Ann. Acad. Sci. Fenn. Ser. A.I. Math.-Phys. No. 50 (1948); MR **10**, 365]. The author sharpens this test by showing that the condition $\prod \mu_n=\infty$ is sufficient even for $F \in O_{AD}{}^0$ and that $O_{AD}{}^0 \subset O_{AD}$. Here $O_{AD}{}^0$ is the class of Riemann surfaces with the following property: on any bordered sub-region G, with border Γ, every nonconstant analytic function with vanishing real part on Γ has an infinite Dirichlet integral over G. The extension from O_{AD} to $O_{AD}{}^0$ also carries over to other criteria.

It is established that a covering surface $F \in O_{AD}{}^0$ with a finite number of sheets above the complex plane has the Iversen property. The class $O_{AB}{}^0$ defined by bounded functions is shown to be a proper sub-class of $O_{AD}{}^0$.
L. Sario (Los Angeles, Calif.)

11120:

Kuramochi, Zenjiro. Correspondence of sets on the boundaries of Riemann surfaces. Proc. Japan Acad. **36** (1960), 112–117.

Given a Riemann surface R of finite genus and positive ideal boundary B, consider a closed subset $B' \subset B$ of positive capacity, and a noncompact bordered subregion G of R. Suppose there exists, for every $z \in B'$, a sector

$S(z) \subset G$ with vertex z. Then the author proves that the double of G with respect to the border is a Riemann surface with positive boundary. If, in a specified sense, every point of the cluster set $H(f(z))$ at B' of a nonconstant analytic function $f(z)$ on R is boundedly covered, then $H(f(z))$ is shown to be of positive logarithmic capacity. Related theorems are given for plane regions.
L. Sario (Los Angeles, Calif.)

11121:

Kuramochi, Zenjiro. Mass distributions on the ideal boundaries. Proc. Japan Acad. **36** (1960), 118–122.

Proof, together with some applications, of a previously announced theorem of the author [Osaka Math. J. **8** (1956), 145–186; MR **19**, 23].
L. Sario (Los Angeles, Calif.)

11122:

Oikawa, Kôtaro. A constant related to harmonic functions. Japan J. Math. **29** (1959), 111–113.

Let K be a compact set on an open Riemann surface W. Consider the family of harmonic functions u on W which are not of constant sign on K. It is known that there exists a constant q, $0<q<1$, independent of u, such that $\max_K |u| \leqq q \cdot \sup_W |u|$ [the reviewer, Trans. Amer. Math. Soc. **72** (1952), 281–295; MR **13**, 735]. The author discusses the evaluation of q. He has earlier shown [Proc. Amer. Math. Soc. **11** (1960), 425–428; MR **22** #5731] that, in terms of the diameter D_K of K in the Poincaré metric, $(2/\pi)$ Arctan sinh $2D_K$ is an admissible value for q; it is the smallest possible value if W is simply connected and $W \notin O_{HB}$. The author now introduces the metric $d\sigma = B(z)|dz|$, where $B(z) = \sup |u_x - iu_y|$ in the family of harmonic functions u on $W \notin O_{HB}$ with $|u| \leqq 1$. He shows that the diameter Δ_K of K in this metric qualifies as q. In case W is contained in a simply connected W^* such that $W^* - W$ is compact in W^* with vanishing capacity, then $(2/\pi)$ Arctan sinh $(\pi\Delta_K/2)$ is the smallest value of q. The question of the smallest value in other cases remains open.
L. Sario (Los Angeles, Calif.)

11123:

Hayashi, Kazumichi. Les solutions positives de l'équation $\Delta u = Pu$ sur une surface de Riemann. Kōdai Math. Sem. Rep. **13** (1961), 20–24.

Let S be an open Riemann surface with a single end, and let P be a smooth non-negative density on S. The author considers classes of solutions of the equation (1) $\Delta u = Pu$ on S. Let S_1 be a compact subregion of S, and let $PP(S-\bar{S}_1)$ be the class of non-negative solutions of (1) in $S-\bar{S}_1$ which vanish on the boundary of S_1. The number of minimal elements in this class is called the elliptic dimension of S. Let Ω be the largest element in $PP(S-\bar{S}_1)$ which is less than one (the elliptic measure of the ideal boundary), and let U_P be the set of those v in $PP(S-\bar{S}_1)$ such that $\int_{\partial S_1} \partial v/\partial n \, ds = 1$. Let P_0 be the class of bounded solutions of (1) on $S-\bar{S}_1$ which are limits of solutions u_m of (1) on compact subregions $S_m - \bar{S}_1$ with $u_m \equiv 0$ on the boundary of S_m. Then for $u \in P_0$, the set of limit values of u/Ω is the set of values of $\int u \, \partial v/\partial n \, ds$ as v ranges over U_P.

As a corollary each u/Ω has a single limit at the ideal boundary if and only if the elliptic dimension of S is one.
H. L. Royden (Stanford, Calif.)

11124:

Gusman, S. Ya. Uniform approximation of continuous functions on Riemann surfaces. Dokl. Akad. Nauk SSSR **130** (1960), 963–965 (Russian); translated as Soviet Math. Dokl. **1**, 105–107.

Generalization of Mergelyan's theorem on approximation by rational functions to functions defined on a closed Riemann surface: Let $\mathscr{R}$ be a closed Riemann surface. The function $f(P)$ defined on a closed set $E \subset \mathscr{R}$ can be expanded into a uniformly convergent series of rational functions on $\mathscr{R}$ whose only pole is at $Q \in \mathbf{C}E$, if and only if $\mathbf{C}E$ is simply connected and $f(P)$ is analytic at all interior points of E and continuous on E.

Generalizations to the approximation of differentials by Abelian differentials with assigned poles are given as well as the following theorem: If the planar measure of the continuum E is zero, then any function continuous on E can be expanded into a uniformly convergent series of rational functions on $\mathscr{R}$. *W. H. J. Fuchs* (Ithaca, N.Y.)

11125a:

Kusunoki, Yukio; Mori, Shin'ichi. On the harmonic boundary of an open Riemann surface. I. Japan. J. Math. **29** (1959), 52–56.

11125b:

Kusunoki, Yukio; Mori, Shin'ichi. On the harmonic boundary of an open Riemann surface. II. Mem. Coll. Sci. Univ. Kyoto Ser. A Math. **33** (1960/61), 209–223.

Let R be an open Riemann surface and Δ the harmonic boundary introduced by the reviewer [*Contributions to the theory of Riemann surfaces*, pp. 107–109, Princeton Univ. Press, Princeton, N.J., 1953; MR **15**, 25]. In these papers, the authors study various properties of Δ, including the significance of the number of points in Δ. New proofs and extensions of results of Kuramochi [Proc. Japan Acad. **31** (1955), 25–30; MR **16**, 1013] and of Constantinescu and Cornea [Nagoya Math. J. **13** (1958), 169–233; MR **20** #3273] are obtained. In the second paper, harmonic measures of sets in Δ are studied and the points of a subset Δ_0 of Δ are shown to be in one-to-one correspondence with the minimal functions in the class $\underline{HD}$ of limits of monotone decreasing sequences of HD functions. *H. L. Royden* (Stanford, Calif.)

11126:

Shimura, Goro. Sur les intégrales attachées aux formes automorphes. J. Math. Soc. Japan **11** (1959), 291–311.

Eichler developed in his paper in Math. Z. **67** (1957), 267–298 [MR **19**, 740] a theory of the periods of integrals associated to automorphic forms in one complex variable, and showed the usefulness of this theory in obtaining formulas for the traces of modular correspondences. In the present paper the author extends this theory by showing that in certain cases the structure of an Abelian variety can also be given to the periods of such integrals. In more detail, let G be a Fuchsian group of automorphisms of the upper half-plane in one complex variable, and $f(z)$ be a cusp form of weight $n+2$ for G, where n is an even integer. Eichler associated to the function $f(z)$ its $(n+1)$-fold iterated indefinite integral $F(z)$, defined

uniquely to within a polynomial of degree n, and then associated to the elements $\sigma \in G$ the periods $x(\sigma) = F(\sigma z)\cdot(d\sigma/dz)^{-n/2} - F(z)$, which are polynomials of degree n defined uniquely to within the obvious equivalence. The equivalence classes of these sets of periods can be interpreted as elements of the first cohomology group of G, with coefficients in the space of polynomials of degree n under the appropriate action of the group G; and the mapping associating to each cusp form its classes of periods is an isomorphism into that cohomology group, the image of the space of cusp forms having half the dimension of the cohomology group [Eichler, loc. cit.]. The present author begins by introducing a convenient paraphrase of the above, essentially considering the coefficients of the polynomials formally as elements of an $(n+1)$-dimensional complex vector space; the periods then appear as elements of the first cohomology group of G with coefficients in an $(n+1)$-dimensional complex vector space, on which the group G acts via a representation corresponding to the above polynomial action. This has the advantage of rewriting the iterated indefinite integrations of Eichler as simple indefinite integrations of vector-valued differential forms, thus exhibiting more explicitly the relations noted by Eichler between this theory and the earlier work of Weil on vector-valued differential forms [J. Math. Pures Appl. (9) **17** (1938), 47–87]. The author then shows that the correspondence associating to the cusp forms the real parts of their periods is actually an isomorphism of the space of cusp forms onto the real cohomology subgroup. If the action of G on the real cohomology group is rational, so that the integral cohomology group is a well-defined lattice, this then induces a lattice into the space of cusp forms, and the quotient space is a compact complex torus. The Petersson inner product is then used to give this torus the structure of an Abelian variety. (Actually the results are more general, since it suffices to have the action of G rational after perhaps an inner automorphism of suitable type.) Some applications to the Hecke correspondences are given in conclusion.

R. C. Gunning (Princeton, N.J.)

11127:

Schöhage, Arnold. Über das Wachstum zusammengesetzter Funktionen. Math. Z. **73** (1960), 22–44.

This article is concerned with the study of the growth of the composition $f(g)$ of two entire functions f and g. An extensive bibliography of previous results on the subject is given. It is observed that these results suffer from the fact that the usual measure of the growth of an entire function, i.e., the order $\rho(f)$, is not delicate enough to throw much light on the relation between the growth of the composition and the growths of the components f and g. The author is thus led to define a new measure of growth of an entire function, namely,

$$\overline{W}_{i,k}(f) = \limsup_{r\to\infty}\left(\frac{\log_{k+i} M(r)}{\log_k r}\right).$$

Here $M(r) = \max_{|z|=r}|f(z)|$, $\log_k$ is the kth iterate of the real logarithm, and i and k are integers. $\underline{W}_{i,k}$ and $W_{i,k}$, in case it exists, are defined analogously. In terms of this notation one has $\overline{W}_{1,1}(f) = \rho(f)$, and in case P_n is a polynomial of nth degree $W_{0,1}(P_n) = n$. The main result of the paper is the so-called "multiplication theorem":

$$\begin{aligned}\overline{W}_{j,k+i-j}(f)\cdot\overline{W}_{i-j,k}(g) &\geqq \overline{W}_{i,k}(f(g))\\ &\geqq \overline{W}_{j,k+i-j}(f)\cdot\underline{W}_{i-j,k}(g)\\ &\geqq \underline{W}_{i,k}(f(g)) \geqq \underline{W}_{j,k+i-j}(f)\cdot\underline{W}_{i-j,k}(g)\end{aligned}$$

and

$$\overline{W}_{i,k}(f(g)) \geqq \underline{W}_{j,k+i-j}(f)\cdot\overline{W}_{i-j,k}(g) \geqq \underline{W}_{i,k}f(g).$$

In case $W_{j,k+i-j}(f)$ and $W_{i-j,k}(g)$ both exist this reduces to

$$W_{i,k}(f(g)) = W_{j,k+i-j}(f)\cdot W_{i-j,k}(g).$$

The proof is based on a theorem of Pólya [J. London Math. Soc. **1** (1926), 12–15]. Applications are made to the solvability in entire functions of the functional equation $f(f(z)) = g(z)$ and related problems.

W. J. Thron (Boulder, Colo.)

11128:

Malliavin, Paul. Fonctions de type exponentiel minimum ayant des zéros donnés. Séminaire P. Lelong, 1958/59, exp. 8, 7 pp. Faculté des Sciences de Paris, 1959.

Preliminary expository report of the author's joint work with the reviewer. The principal result described gives necessary and sufficient conditions that an arbitrary given sequence $0 < \lambda_0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots$ may be among the zeros of an entire function $f(z)$ of exponential type, $f(z)$ not identically zero, with $|f(iy)| \leqq \exp(\pi|y|)$. The version given in this report is now obsolete as a result of improvements and simplifications obtained since.

L. A. Rubel (New York)

11129:

Benevolenskiĭ, V. I. Some limit properties of entire functions of finite order and their derivatives. Issledovaniya po sovremennym problemam teorii funkciĭ kompleksnogo peremennogo, pp. 166–175. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

Let $f(z)$ be an entire function of finite order $\rho > 0$ and normal type. The author studies the relations between the indicator function

$$h(\varphi) = \limsup_{r\to\infty}\frac{\log|f(re^{i\varphi})|}{r^\rho}$$

of $f(z)$ and that of $f'(z)$, denoted by $h_1(\varphi)$. It is shown that if, for some value of φ, $0 \leqq \varphi < 2\pi$, $h(\varphi) \neq 0$, then $h_1(\varphi) = h(\varphi)$. Further, if $h(\varphi) = 0$ on some segment $[\varphi_1, \varphi_2]$, then $h_1(\varphi) \leqq 0$ on the same segment. Some corollaries of these results are given. It is pointed out that the above theorems remain valid if the indicator function is generalized by the introduction of proximate orders [G. Valiron, *Lectures on the general theory of integral functions*, Édouard Privat, Toulouse, 1923]. The second part of the paper is concerned with various applications of these theorems. In particular, it is proved that if $f(z)$ is an entire function of finite order which tends to the same finite value as $z\to\infty$ along every ray for which $|\arg z - \theta| < \delta$, $\delta > 0$, $0 \leqq \theta < 2\pi$, then $f'(z)$ tends to zero as $z\to\infty$ on these rays. {It should be noted that a sharper result follows if one applies Rogosinski's theorem [J. London Math. Soc. **20** (1945), 100–109; MR **8**, 324].}

W. Seidel (Detroit, Mich.)

11130:

Brudnyĭ, Yu. A.; Gopengauz, I. E. Generalization of a theorem of Hardy and Littlewood. Mat. Sb. (N.S.) **52** (94) (1960), 891–894. (Russian)

Let $f(z)$ be a function holomorphic in $|z|<1$ and continuous on $|z|\leqq 1$. The modulus of continuity $\omega(\delta)$ of $f(z)$ on $|z|=1$ is defined by the relation

$$\omega(\delta) = \sup_{|\theta_1-\theta_2|\leqq\delta} |f(e^{i\theta_1})-f(e^{i\theta_2})|.$$

A result, due to Hardy and Littlewood [Math. Z. **34** (1931), 403–439], concerning such functions is generalized and a theorem of Geronimus [Mat. Sb. (N.S.) **38 (80)** (1956), 319–330; MR **18**, 386] is sharpened by the authors. It is shown that in order that $f(z)$ have the modulus of continuity $\omega(\delta)$ on $|z|=1$, it is necessary that the inequality

$$(1)\qquad |f'(z)| \leqq C\frac{\omega(1-|z|)}{1-|z|},$$

where C is a constant independent of ω, hold in $|z|<1$. Furthermore, if

$$(2)\qquad \int_0^x \frac{\omega(t)}{t}\,dt = O(\omega(x))$$

{there appears to be a misprint in the paper at this point}, then (1) implies that $f(z)$ has a modulus of continuity not exceeding $\omega(\delta)$ on $|z|=1$. {Here the authors state instead that $f(z)$ has the modulus of continuity $\omega(\delta)$, which is not what is actually proved.} Moreover, if $f(z)$ has the modulus of continuity $\omega(\delta)$ on $|z|=1$ satisfying condition (2), then $|f(z_1)-f(z_2)|\leqq C\omega(|z_1-z_2|)$ for any pair of points z_1, z_2 in the closed disk $|z|\leqq 1$. An example is given to show that condition (2) cannot, in general, be dropped. A remark is made to the effect that the above results remain valid if formulated with L^p norm, for $p\geqq 1$.
W. Seidel (Detroit, Mich.)

11131:

Tsuji, Masatsugu. A theorem on the boundary behaviour of a meromorphic function in $|z|<1$. Comment. Math. Univ. St. Paul. 8 (1960), 53–55.

Let $w(z)$ be a meromorphic function in $|z|<1$ satisfying the condition

$$\int_0^{2\pi} \frac{|w'(re^{i\theta})|}{1+|w(re^{i\theta})|^2}\,d\theta \leqq K$$

for $0\leqq r<1$, so that every circle $|z|=r$ is mapped on a curve of length $\leqq K$ on the w-sphere. Denoting by $L(\theta)$ the radius of $|z|<1$ through $e^{i\theta}$ and by $L_\alpha(e^{i\theta})$ $(-\frac{1}{2}\pi<\alpha<\frac{1}{2}\pi)$ the segment in $|z|<1$ which passes through $e^{i\theta}$ and makes an angle α with $L(\theta)$, it is shown that for almost all θ, $L(\theta)$ is mapped on a rectifiable curve on the w-sphere. Moreover, there exists a set E of measure 2π on $|z|=1$, such that if $e^{i\theta}\in E$, then for almost all α, $L_\alpha(e^{i\theta})$ is mapped on a rectifiable curve. For all such α, $\lim_{z\to e^{i\theta}} w(z)=w(e^{i\theta})$, $z\in L_\alpha(e^{i\theta})$, exists and is independent of α. For all other values of α, $L_\alpha(e^{i\theta})$ is a Julia direction of $w(z)$, i.e., $w(z)$ assumes every value, except at most two, infinitely often in every angular domain with vertex at $e^{i\theta}$ containing $L_\alpha(e^{i\theta})$. {Reviewer's remark: It would be of interest to know whether such Julia directions actually exist.}
W. Seidel (Detriot, Mich.)

11132:

Bonami, A. A. Boundary properties of functions regular in a strip. Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, pp. 95–110. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

The author begins by deriving a representation for functions that are positive and harmonic in a strip [discussed independently by Widder, Proc. Amer. Math. Soc. **12** (1961), 67–72]. Class D consists of functions f analytic in the strip, such that the least harmonic majorant of $\log|f(z)|$ is the uniform limit from below of harmonic functions that are bounded above. Let $b(z)$ denote the analogue of a Blaschke product over the zeros of $f(z)$. Then a necessary and sufficient condition for f to belong to D in the strip $0<x<\pi$ is that there are x_1 and x_2 in $(0,\pi)$ for which

$$\lim_{y\to\infty} e^{-y}\log^+|b(x_1+iy)| = \lim_{y\to-\infty} e^{y}\log^+|b(x_2+iy)| = 0,$$

$$\lim_{y\to\infty} e^{-y}\log^+|f(x_1+iy)| = \lim_{y\to-\infty} e^{y}\log^+|f(x_2+iy)| = 0,$$

and that for all y_1 and y_2, with $\xi=0$ or π,

$$\lim_{x\to\xi}\int_{y_1}^{y_2}\log^+|f(x+i\eta)|\,d\eta = \int_{y_1}^{y_2}\log^+|f(\xi+i\eta)|\,d\eta = 0.$$

The author then generalizes results of Hardy, Ingham and Pólya [Proc. Roy. Soc. London Ser. A **113** (1927), 542–569] on the convexity of mean values of functions analytic in a strip, both by weakening the hypotheses of growth imposed on the function and by considering means of the form $\limsup y^{-\mu}\int_{-y}^{y}|f(x+i\eta)|^p dy$. Similar results are obtained for subharmonic functions of class LP (non-negative with subharmonic logarithm)..
R. P. Boas, Jr. (Evanston, Ill.)

11133:

Hummel, J. A. Extremal problems in the class of starlike functions. Proc. Amer. Math. Soc. **11** (1960), 741–749.

In an earlier paper [same Proc. **9** (1958), 82–87; MR **20** #1779], the author developed a variational method for the class S^* of analytic functions which are regular and starlike in $|z|<1$, and normalized by $f(0)=0$, $f'(0)=1$. In the present paper he studies the classes of extremal problems to which his method can be applied. His main result is that the problem $\mathrm{Re}\,\{J[f(z)]\}=\max$, $f(z)\in S^*$, can be solved by his method if the complex-valued functional $J[f]$ is "linear in the small", i.e., if $J[f+\varepsilon g]=J[f]+\varepsilon J_1[f,g]+o(\varepsilon)$, where the functional $J_1[f,g]$ is linear in g. The extremal function $f(z)$ is shown to be a solution of the differential equation $zf'R(z)=fQ(z)$, where $R(z)$ and $Q(z)$ can be expressed in terms of $J_1[f(z),(z+z_0)(z-z_0)^{-1}]$. The author also shows that, in general, the extremal functions map the unit disk onto domains bounded by radial slits.
Z. Nehari (Pittsburgh, Pa.)

11134:

Alenicyn, Yu. E. Functions without common values. Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, pp. 34–38. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

A number of results are stated, without proof, concerning inequalities satisfied by $f_k(z)$, $f_k'(z)$ $(k=1,2)$ when $f_1(z)$, $f_2(z)$ are functions meromorphic in a region B of the z-plane and $f_1(z_1)\neq f_2(z_2)$ for z_1, z_2 in B. In particular, best possible estimates are given for $|f_1'(z_1)f_2'(z_2)|$. Special results are obtained for the unit circle and for an annulus.

Applications are given to functions $f(z)$, analytic in B, such that $f(a)=0$ for some fixed a in B, and such that $f(z_1)f(z_2)\neq 1$ for z_1, z_2 in B. Relations with previous work of Ahlfors, Jenkins, Robinson and the author are pointed out.
W. Kaplan (Ann Arbor, Mich.)

11135:

Pogorzelski, W. Propriétés d'une classe de fonctions holomorphes aux fonctions limites discontinues. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 711–714. (Russian summary, unbound insert)

Let S^+ denote a domain bounded by simple closed curves $L_0, L_1, \cdots, L_n$, and let $S_1^-, \cdots, S_n^-$ denote the bounded, distinct domains complementary to S^+ determined by $L_1, \cdots, L_n$, while S_0^- denotes the exterior (and possibly unbounded) domain determined by L_0; the possibility that L_0 is absent is not excluded. The author makes a slight modification of the class $\mathscr{H}_\alpha{}^h(c_{k_1}, \cdots, c_{k_s})$ of discontinuous complex functions of an earlier paper [same Bull. **7** (1959), 311–317; MR **22** #763] and announces some properties of the class $H_\alpha{}^h$ of analytic functions having limiting functions in the class $\mathscr{H}_\alpha{}^h$.
A. J. Lohwater (Providence, R.I.)

11136:

Tumarkin, G. C.; Havinson, S. Ya. Qualitative properties of solutions of certain types of extremal problems. Issledovaniya po sovremennym problemam teorii funkciĭ kompleksnogo peremennogo, pp. 77–95. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

The paper starts with a short historical introduction and brings a condensed listing of results obtained by the authors in a previous paper [Mat. Sb. (N.S.) **46** (**88**) (1958), 195–228; see MR **21** #3545 and the bibliography mentioned there]. Further developments of these results are given in eight theorems without proofs.
B. A. Amirà (Jerusalem)

11137:

Lewandowski, Zdzislaw. Sur certaines classes de fonctions univalentes dans le cercle-unité. Ann. Univ. Mariae Curie-Skłodowska Sect. A **13** (1959), 115–126. (Polish and Russian summaries)

Let Σ_0 denote the class of functions $f(z)$ regular and univalent in $|z|<1$ for which $f(0)=0$. Let S denote the subclass of Σ_0 for which $f'(0)=1$, $S_0\subset\Sigma_0$ for which $f(z_0)=z_0$ $(0<|z_0|<1)$, $S_p\subset\Sigma_0$ for which $f^{(p)}(z_0)=1$, $p=1, 2, \cdots$. Let S^*, S_0^*, S_p^* denote the corresponding classes of functions which are also star-like with respect to the origin in $|z|<1$.

The author establishes several covering theorems for these classes. The map of $|z|<1$ by $w=f(z)\in S_0$ covers the circle $|w|<\frac{1}{4}(1-|z_0|)^2$. The result is sharp. The theorem was obtained earlier for the subclass S_0^* by W. Rogosinski [Compositio Math. **3** (1936), 199–226]. The author obtains an analogous theorem for the class S_1, which in turn is a generalization of a result due to M. Biernacki for the subclass S_1^* [Mathematica (Cluj) **16** (1940), 44–49; MR **2**, 84]. With the assumption that $|a_n|\leq n$ for $n=2, 3, \cdots, p$ for every $f(z)=z+\sum_2^\infty a_nz^n$ of class S, then if $f\in S_p$, $w=f(z)$ covers

$$|w|<\frac{1}{4(p!)}\frac{(1-|z_0|)^{p+2}}{(p+|z_0|)}$$

and this result is sharp. Finally, if $f(z)\in S^*$, the author obtains

$$\text{(A)}\qquad \left|\left(\frac{z}{f(z)}\right)^{1/2}-\frac{1-z\bar{z}_0}{1-|z_0|^2}\right|\leq\frac{|z-z_0|}{1-|z_0|^2}$$

The particular case $z_0=0$ in (A) was obtained by A. Marx [Math. Ann. **107** (1932), 40–67].
M. S. Robertson (New Brunswick, N.J.)

11138:

Krzyż, Barbara. Sur les fonctions en moyenne (ψ) p-valentes. Ann. Univ. Mariae Curie-Skłodowska Sect. A **12** (1958), 38–44. (Polish and Russian summaries)

Let ψ be a non-decreasing real-valued function on $[0,\infty)$ with $\psi(0)=0$. A function f analytic in $|z|<1$ is said to be mean $(\psi)p$-valent if $\int_0^R p(r)d\psi(r)\leq p\psi(R)$, where $p(r)$ is $(2\pi)^{-1}\int_0^{2\pi} n(re^{i\theta})d\theta$, with $n(w)$ expressing the number of times f assumes the value w in $|z|<1$. The author seeks conditions on a pair of functions ψ and φ such that each mean $(\psi)p$-valent function will also be a mean $(\varphi)p$-valent function and establishes the following theorem.

Let ψ and φ be of class C^1 with ψ' and φ' non-negative and φ'/ψ' of bounded variation in each finite interval. Then a necessary and sufficient condition that each $(\psi)p$-valent function shall also be $(\varphi)p$-valent is that φ'/ψ' be a non-increasing function.
H. L. Royden (Stanford, Calif.)

11139:

Delange, Hubert. Sur les représentations exponentielles des fonctions holomorphes dans le demi-plan supérieur avec une partie imaginaire positive. Séminaire P. Lelong, 1958/59, exp. 7, 10 pp. Faculté des Sciences de Paris, 1959.

Résumé of the paper by Aronszajn and Donoghue, J. Analyse Math. **5** (1956/57), 321–388.

11140:

Walsh, J. L. Note on degree of approximation by bounded analytic functions: Problem β. Trans. Amer. Math. Soc. **96** (1960), 246–258.

Dans un article antérieur [mêmes Trans. **87** (1958), 467–484; MR **20** #3298], l'auteur traitait du problème suivant (problème β): soit D une région bornée du plan, et B une partie de sa frontière; soit $f(z)$ une fonction continue sur B, et analytique au voisinage de B dans D; soit enfin $f_n(z)$ une suite de fonctions analytiques et bornées dans D, continues sur B, et qui convergent sur B vers $f(z)$; quelle relation y-a-t-il entre les propriétés de $f(z)$, de $X_n=\sup_{z\in D}|f_n(z)|$ et de $Y_n=\sup_{z\in B}|f_n(z)-f(z)|$? Le théorème 1 de l'article cité faisait intervenir une condition de Lipschitz satisfaite par $f(z)$ sur une courbe voisine de B; le but du présent article est de la remplacer par une condition de Lipschitz en moyenne (théorème 6). Pour cela, l'auteur utilise les notions qu'il a introduites avec Russell [ibid. **92** (1959), 355–370; MR **21** #7311]. La classe $H(k,\alpha,p)$ (k entier, $0<\alpha<1$, $p\geq 1$) sur le cercle γ: $|z|=1$ est l'ensemble des fonctions f holomorphes dans $|z|<1$, et satisfaisant

$$\int_\gamma |f^{(k)}(ze^{ih})-f^{(k)}(z)|^p\,dz = O(h^{\alpha p}),\quad h\to 0,\ \text{si } k\geq 0,$$

$$\int_\gamma |f(rz)|^p|dz| = O((1-r)^{k+\alpha}),\quad r\to 1,\ \text{si } k<0.$$

La classe $H'(k, \alpha, p)$ sur γ est formée des sommes d'une fonction de la classe $H(k, \alpha, p)$ et d'une function analytique au voisinage de γ. Si J est une courbe de Jordan analytique, on définit de façon naturelle les classes $H(k, \alpha, p)$ et $H'(k, \alpha, p)$ sur J à l'aide des transformations conformes de l'intérieur de γ en l'intérieur de J. Un résultat simpie et important (théorème 4) est qu'une transformation conforme biunivoque appliquant J sur J laisse invariante la classe $H'(k, \alpha, p)$ sur J. *J.-P. Kahane* (Montpellier)

11141:

Naftalevič, A. G. On interpolation by functions of bounded characteristic. Vilniaus Valst. Univ. Mokslų Darbai. Mat. Fiz. Chem. Mokslų Ser. **5** (1956), 5–27. (Russian)

The author considers the following problem. Let n be a fixed positive integer and let

$$R_k(z) = \frac{a_{k,n}}{(z-z_k)^n} + \frac{a_{k,n-1}}{(z-z_k)^{n-1}} + \cdots + \frac{a_{k,1}}{(z-z_k)}$$

($k=1, 2, \cdots$, $|z_k|<1$, $\lim |z_k|=1$) be given rational functions. Under what conditions on the moduli $|a_{k,j}|$ and on the points z_k will there exist a meromorphic function of bounded characteristic in the unit circle with the principal parts $R_k(z)$ and no other singularities? He obtains the following results on this interpolation problem.

Theorem 1: Let

$$(1) \qquad \limsup\,(1-|z_i|)\log^+|a_{ij}| < \infty \qquad (j = 1, 2, \cdots, n).$$

Then the interpolation problem can be solved if and only if (a) $\sum(1-|z_i|)<\infty$ and (b) the points z_i lie inside a polygon inscribed in the unit circle.

Furthermore, if condition (1) is not satisfied then one can find numbers a_{ij}', $|a_{ij}'| \leqq |a_{ij}|$, for which the interpolation problem cannot be solved.

Theorem 2: Let $\sum(1-|z_i|)<\infty$. Then the condition

$$(2) \qquad \sum(1-|z_i|)\log^+\frac{|A_i|}{(1-|z_i|)^2} < \infty$$

is sufficient for the existence of a function of bounded characteristic with principal parts $A_i/(z-z_i)$.

Furthermore, if condition (2) is not satisfied, then there exist points z_k', $|z_k'|=|z_k|$, for which this interpolation problem cannot be solved.

The author also constructs a class of Blaschke products whose derivatives do not have bounded characteristic.

The last section deals with analytic functions of bounded characteristic. The author's interpolation problem now takes the form of prescribing the first n terms in the Taylor expansion of f about each of the points z_k:

$$(3) \qquad a_{k,0}+a_{k,1}(z-z_k)+\cdots+a_{k,n-1}(z-z_k)^{n-1} \qquad (k = 1, 2, \cdots).$$

Theorem 3: Let the numbers a_{ij} satisfy condition (1). Then the interpolation problem can be solved if and only if (a) and (b) of Theorem 1 hold and in addition (c) $\limsup\,(1-|z_k|)\log^+|\gamma_{kj}|<\infty$ ($j=1, \cdots, n-1$). Here the numbers γ_{kj} ($k=1, 2, \cdots; j=1, \cdots, n-1$) are the first n terms in the Taylor expansion of the function $(z-z_k)^n/(B(z))^n$ about the point z_k. $B(z)$ is the Blaschke product formed over all the points z_k.

The author shows that, given any sequence z_k for which $\sum(1-|z_k|)<\infty$, there is a sequence z_k' with $|z_k'|=|z_k|$, for which $|B'(z_k)|(1-|z_k|)>c>0$. {This condition is precisely that given later by Lennart Carleson [Amer. J. Math. **80** (1958), 921–930; MR **22** #8129] for the solvability of an interpolation problem for bounded analytic functions.} *A. L. Shields* (New York)

11142:

Temlyakov, A. A. Integral representations. Dokl. Akad. Nauk SSSR **131** (1960), 263–264 (Russian); translated as Soviet Math. Dokl. **1**, 244–245.

The integral formulas given by Temlyakov in same Dokl. **120** (1958), 976–979 [MR **20** #5884] for an analytic function of two complex variables, which is continuous with continuous first partial derivatives, hold for the domain bounded by the three hypersurfaces $0 \leqq |w| \leqq r_1(0)$, $|z|=r_2(0)$; $|w|=r_1(\tau)$, $|z|=r_2(\tau)$, $0<\tau<1$; $|w|=r_1(1)$, $0\leqq|z|\leqq r_2(1)$. For the properties of r_1 and r_2 see the review cited above. Here it is assumed that $r_1(0)>0$, $r_2(1)>0$. *J. Mitchell* (University Park, Pa.)

11143:

Hitotumatu, Sin. Note on the Picard theorem in the case of several variables. Japan. J. Math. **29** (1959), 1–4.

The author wants to generalize Picard's theorem. The reviewer believes that Theorem 1 has to be restated as follows: Let $f(z_1, \cdots, z_n, w)$ be holomorphic in $|z_1|<r, \cdots$, $|z_n|<r$, $|w|<r$, having essential singularities on $w=0$. Then either f assumes every value in each neighborhood of the origin or else there is a subset A of $w=0$ whose closure contains the origin that has the property that for each point $(z_1^0, \cdots, z_n^0, 0)\in A$ and each value a, except one, the equation $f(z_1^0, \cdots, z_n^0, w)=a$ has infinitely many solutions. Similarly, the rest of the statements must be rephrased. *H. Röhrl* (Minneapolis, Minn.)

11144:

Bochner, S. Classes of holomorphic functions of several variables in circular domains. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 721–723.

Let C^k be the space of k-tuples of complex numbers $\mathbf{z}=(z_1, \cdots, z_k)$. Let D be a "bounded circular domain of C^k with origin", i.e., such that $\mathbf{a}\in D$ and $|z|\leqq 1 \Rightarrow z\mathbf{a}\in D$. Let ω be a Borel measure on B, the boundary of D, which is "circularly invariant", i.e., such that B_0 is a measurable subset of $B \Rightarrow \omega(B_0)=\omega(e^{i\theta}B_0)$ for all real θ. The following is stated. Theorem 3: D, B, ω being as above, let f be holomorphic on D and in "the Hardy class H_λ, $\lambda>0$, with respect to the measure ω", i.e., such that

$$\sup_{0<r<1}\int_B |f(r\zeta)|^\lambda\, d\omega(\zeta) = C^\lambda < \infty.$$

Then

$$\int_B \sup_{0<r<1} |f(r\zeta)|^\lambda\, d\omega(\zeta) \leqq a_\lambda C^\lambda;$$

also there exists a function F on B, integrable with respect to ω, such that

$$\int_B |f(r\zeta)-F(\zeta)|^\lambda\, d\omega(\zeta) \to 0, \quad \text{as } r\to 1-.$$

In particular D can be a sphere in C^k and ω the ordinary $(2k-1)$-dimensional surface area; or D can be the polycylinder $P_k=[|z_1|<1, \cdots, |z_k|<1]$, and ω concentrated

on the k-dimensional subset $[|z_1|=1, \cdots, |z_k|=1]$ of its total $(2k-1)$-dimensional boundary. The special case $\lambda=1$, $D=P_k$ and $d\omega=d\theta_1\cdots d\theta_k$ was established earlier by the author in Ann. of Math. (2) **45** (1944), 708–722 [MR **6**, 124] along with other powerful results.

The author claims that Theorem 3 is derivable by first extending its (quite classical) form for $k=1$ and $D=[|z|<1]$ (Theorem 1) to functions $\phi(\ ,\xi)$, again of one complex variable, but depending on a parameter ξ which varies over a measure space X, and such that ϕ is measurable on $[|z|<1]\cap X$, and for almost all $\xi\in X$, $\phi(\ ,\xi)$ is holomorphic on $[|z|<1]$, and

$$\sup_{0<r<1}\int_X d\xi\int_0^1|\phi(re^{2\pi i\theta},\xi)|^\lambda\,d\theta = C^\lambda<\infty.$$

This extension (Theorem 2) is expected to yield Theorem 3.

For $\lambda>1$, the counterpart of Theorem 3 for the real part u of f is stated, and for $\lambda=1$ an analogous result under the condition

$$\sup_{0<r<1}\int_B|u(r\zeta)|\log^+|u(r\zeta)|\,d\omega(\zeta)=C<\infty$$

(Theorem 4). Also for $\lambda>1$, the author states an extension of M. Riesz's theorem on the inequality between the Hardy-class norms of the real and imaginary parts u, v of f (Theorem 5). *P. Masani* (Bloomington, Ind.)

11145:

Hua, L. K.; Look, K. H. Theory of harmonic functions of classical domains. I. Harmonic functions in the hyperbolic space of matrices. Acta Math. Sinica **8** (1958), 531–547. (Chinese. English summary)

Contained in #11148.

11146:

Hua, L. K.; Look, K. H. Theory of harmonic functions of the classical domains. II. Harmonic functions in the hyperbolic space of symmetric matrices. Acta Math. Sinica **9** (1959), 295–305. (Chinese. English summary)

11147:

Hua, L. K.; Look, K. H. Theory of harmonic functions of the classical domains. III. Harmonic functions in the hyperbolic space of skew symmetric matrices. Acta Math. Sinica **9** (1959), 306–314. (Chinese. English summary)

11148:

Hua, L. K.; Look, K. H. Theory of harmonic functions in classical domains. Sci. Sinica **8** (1959), 1031–1094.

The authors continue their study of the four classical types of bounded symmetric domains in several complex variables. The main results in this paper are the determination of the boundary structure of these domains and the description of the boundary behaviour of the solutions of the Dirichlet problem. The methods consist in explicit computations in each of the four cases; the exceptional domains are not considered at all. In somewhat more abstract terms the main results could be summarized as follows. Let $D=G/K$, an irreducible classical domain, imbedded in C^n in the natural way [cf. Harish-Chandra, Amer. J. Math. **78** (1956), 564–628; MR **18**, 490]. The boundary of D decomposes into r orbits of G (r the rank of the symmetric space G/K: $B_1, \cdots, B_r$, with $B_{j+1}\subset\bar B_j$ $(j=1,\cdots,r-1)$; each B_j $(j=1,\cdots,r-1)$ is a direct product space $D_j\times L_j$, where $D_j=G_j/K_j$ is a classical symmetric domain with $G_j\subset G$ and L_j is a homogeneous space of K). B_r is a homogeneous space of K, and is the Šilov boundary of D. All these homogeneous spaces are determined explicitly for all the four classical types as spaces of matrices. Now let there be given a real-valued continuous function φ on the Šilov boundary B_r of D. It is known from the work of J. Mitchell, K. Morita, D. Lowdenslager that the Dirichlet problem for B_r can be solved in the sense that there exists a unique function f in D satisfying the Laplace-Beltrami equation for the invariant metric of D and having the continuous boundary values φ on B_r. Explicit Poisson formulas are known in each of the four classical cases. The authors here study the behaviour of f on the entire boundary. It turns out that it always has continuous boundary values everywhere, and on each $B_j=D_j\times L_j$ $(j=1,\cdots,r-1)$ its restriction to any cross-section above D_j projects into a harmonic function on D_j. The proof is based on a study of the Poisson formulas. In the last sections the authors indicate some applications and extensions of the above ideas. Using the fact that the group U_n of $n\times n$ unitary matrices appears as the Šilov boundary of one of the classical domains, they define the Abel summation of Fourier series on U_n, and prove that every continuous function on U_n is represented in this sense by its Fourier series. They also give an example of a domain in real Euclidean space, the domain consisting of $m\times n$ real matrices X such that $I-XX'>0$, which is a symmetric space without a complex structure, and for which they can prove results very similar to the preceding ones [cf. also F. I. Karpelevič, Dokl. Akad. Nauk SSSR **124** (1959), 1199–1202; MR **21** #3013]. *A. Korányi* (Berkeley, Calif.)

SPECIAL FUNCTIONS

See also B11527.

11149:

MacRobert, Thomas M. Fourier series for E-functions. Math Z. **75** (1960/61), 79–82.

If $0\leqq\theta\leqq\pi$ and $|\arg z|<\pi$, then

$$\Gamma(3/2)\sin\theta E(p;\alpha_r:q;\rho_s:z/\sin^2\theta)=\sum_0^\infty z^{-n}E\begin{pmatrix}n+3/2,\ \alpha_1+n,\ \cdots,\ \alpha_p+n:z\\ 2n+2,\ \rho_1+n,\ \cdots,\ \rho_q+n\end{pmatrix}\sin(2n+1)\theta.$$

N. D. Kazarinoff (Ann Arbor, Mich.)

11150:

Fettis, Henry E. Note on $\int_0^\infty e^{-z}J_0(\eta x/\xi)J_1(x/\xi)x^{-n}\,dx$. Math. Comp. **14** (1960), 372–374.

The author claims that the closed expression in terms of elliptic integrals which is given in P. F. Byrd and M. D. Friedman's *Handbook of elliptic integrals for engineers and physicists* [Springer, Berlin, 1954; MR **15**, 702] for the function (1) $\int_0^\infty e^{-pt}J_0(rt)J_1(st)dt$ is incorrect, first, because the formula fails numerical checks; second, because it does not satisfy known relations in special

cases. He then derives another expression for (1), in terms of complete elliptic integrals of the first and third kinds, but does not indicate whether this has been tested numerically. *F. W. J. Olver* (Washington, D.C.)

11151:

MacRobert, T. M. Beta-function formulae and integrals involving E-functions. Math. Ann. **142** (1960/61), 450–452.

An augmented version of the Beta function integral is established, viz., $\mathrm{Re}\{\alpha\}>0$, $\mathrm{Re}\{\beta\}>0$,

$$(1)\quad \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\{1+ct+d(1-t)\}^{-\alpha-\beta}\,dt = (1+c)^{-\alpha}(1+d)^{-\beta}B(\alpha,\beta).$$

The author shows that if the integrand of (1) is multiplied by an E-function (with appropriate arguments) the resulting integral can be evaluated in terms of other E-functions. *T. Erber* (Chicago, Ill.)

11152:

Nørlund, Niels Erik. Sur la convergence de certaines séries de facultés. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 532–538.

Let $s=1,\cdots,n$, let α_s and γ_s be complex numbers, $\beta_n=n-1-\sum(\alpha_s+\gamma_s)$, and x a complex variable. By studying solutions of an appropriate hypergeometric differential equation the author derives an inverse factorial expansion of

$$\frac{\Gamma(x+\alpha_n+\beta_n+1)}{\Gamma(x-\gamma_n+1)}\prod_{t=1}^{n-1}\frac{\Gamma(x+\alpha_t)}{\Gamma(x-\gamma_t+1)}$$

convergent when $\mathrm{Re}(x+\alpha_t)>0$, $t=1,\cdots,n-1$. He now sets $n=m+p$, specializes $\alpha_{m+1},\cdots,\alpha_n$, and obtains an inverse factorial expansion of

$$(*)\qquad \prod_{r=1}^{m}\Gamma(x+\alpha_r)\prod_{r=1}^{p}\Gamma\left(x+\frac{\beta+r}{p}\right)\Big/\prod_{s=1}^{n}\Gamma(x-\gamma_s+1).$$

If $m>0$, $p<6$, the series is absolutely convergent for $\mathrm{Re}(x+\alpha_r)>0$, $r=1,\cdots,m$; if $m>0$ and $p=6$, absolute convergence prevails for $\mathrm{Re}(x+\alpha_r)>0$ and $\mathrm{Re}(6x+\beta)>0$; and if $p>6$ the series is divergent and represents (*) asymptotically for $|\arg x|<\pi/2$. The case $m=0$ is also analyzed, and the paper contains some related developments. *A. Erdélyi* (Pasadena, Calif.)

11153:

Carlitz, L. The product of basic Bessel functions. Quart. J. Math. Oxford Ser. (2) **11** (1960), 181–186.

The basic analogues of the Bessel functions that are discussed in this paper are

$$I_n(t)=\sum_{r=0}^{\infty}\frac{t^{n+2r}}{(q)_r(q)_{n+r}},$$

$$I_n(t)=\sum_{r=0}^{\infty}\frac{t^{n+2r}}{(q)_r(q)_{n+r}}\,q^{[r(r-1)+(n+r)(n+r-1)]/2},$$

where $(q)_0=1$, $(q)_r=(1-q)(1-q^2)\cdots(1-q^r)$ $(r\geqq 1)$. For these functions the author derives numerous formulas, some of them involving $\sum_{-\infty}^{\infty}x^nI_n(t)I_n(z)$ and similar sums. *A. Erdélyi* (Pasadena, Calif.)

11154:

Sneddon, Ian N. On some infinite series involving the zeros of Bessel functions of the first kind. Proc. Glasgow Math. Assoc. **4**, 144–156 (1960).

A simple and systematic method is given for summing series involving the zeros $\gamma_{\nu,n}$ $(n=1,2,3,\cdots)$ of $J_\nu(z)$. The following series are considered: $\sum_{n=1}^{\infty}\gamma_{\nu,n}^{-2m}$ and $\sum_{n=1}^{\infty}\gamma_{\nu,n}^{-2m+1}J_{\nu+1}^{-1}(\gamma_{\nu,n})$, where m is a positive integer. The method depends on simple properties of Fourier-Bessel series and Dini series. It is capable of obtaining more complicated sums. *H. A. Lauwerier* (Amsterdam)

11155:

Dempsey, E.; Benson, G. C. Note on the asymptotic expansion of the modified Bessel function of the second kind. Math. Comp. **14** (1960), 362–365.

Expressions are given for additional terms in the Dingle [Proc. Roy. Soc. London Ser. A **249** (1959), 270–283; MR **21** #2148a] and (equivalent) Burnett [Proc. Cambridge Philos. Soc. **26** (1930), 145–151] expansions for the converging factor in the asymptotic expansion for the modified Bessel function $K_n(z)$ for large $|z|$. Graphs of the residual error plotted against real positive values of z are also included. *F. W. J. Olver* (Washington, D.C.)

11156:

Cima, Joseph A. Note on a theorem of Grosswald. Trans. Amer. Math. Soc. **99** (1961), 60–61.

The author points out an error in the reviewer's proof [same Trans. **71** (1951), 197–210; MR **14**, 747] of the theorem stating that Bessel polynomials of even degrees have no real zeros [see Burchnall, Canad. J. Math. **3** (1951), 62–68; MR **12**, 499]. He supplies a simple, correct proof. *E. Grosswald* (Philadelphia, Pa.)

11157:

Magnus, Wilhelm; Kotin, Leon. The zeros of the Hankel function as a function of its order. Numer. Math. **2** (1960), 228–244.

This paper contains a detailed discussion of the behaviour of $H_\nu^{(1)}(z)$ for fixed $z=x+iy$ as a function of $\nu=\alpha+i\beta$. Only some of the numerous results can be listed in this review. (2.1) There is an infinite number of zeros if $0\leqq\arg z\leqq\pi$. (2.3) The zeros are simple if $x>0$, $y\geqq 0$. (3.1) Given $\beta\neq 0$, $i\beta$ is a zero for some $z=iy$, $y>0$. (3.2) If ν is a zero for $z=a\nu$ with $a>0$, $\beta>0$, then $a<1$. (3.3) If $y>0$ and ν is a zero then $\mathrm{sgn}(\alpha\beta-xy)=\mathrm{sgn}\,x$. (3.4) If $|x|\leqq y$ and ν is a zero, then $\alpha^2-\beta^2<x^2-y^2$. (4.1.2) If $x>0$, $y=0$ and ν is a zero, then $\alpha\beta>0$ and $|\alpha|>x-(x/2)^{1/3}$. (4.3) If also $|\alpha|<1/2$ then $x<\beta\tan\alpha\pi+1/4$. (4.4) If $x>0$, $y=0$, ν is a zero, $\alpha=\pm 1/2$, then $|\beta|>x/2$. (6.1.2) If $x>0$, $y=0$, $\nu=\nu(x)$ is a zero then $\nu^{-1}\alpha\log x\to 0$, $\nu\to 0$, and $\beta\log x+k\pi=O(\nu x)$ for some integer k, as $x\to 0$. Results not given in this review concern, inter alia, quantitative estimates, the derivative of $\nu=\nu(x)$ with respect to x, behaviour of large zeros, and the growth of $H_\nu^{(1)}(z)$ as a function of ν. *A. Erdélyi* (Pasadena, Calif.)

11158:

Karmazina, L. N. Asymptotic formulas for the functions. $P_{-\frac{1}{2}+i\tau}(x)$ as $\tau\to\infty$. Vyčisl. Mat. **6** (1960), 3–16. (Russian)

The author derives four terms of the asymptotic expansion of spherical Legendre functions in terms of Bessel functions $J_n(x)$, $n=0$, 1, 2, 4. The following integral form of Legendre functions is used to obtain the expansion:

$$P_{-\frac{1}{2}+i\tau}(x) = P_{-\frac{1}{2}+i\tau}(\operatorname{ch}\theta) = \frac{2}{\pi}\int_0^\infty \frac{\cos\tau t}{\sqrt{(2(\operatorname{ch}\theta-\operatorname{ch}t))}}\,dt.$$

Tables of the coefficients of the expansion are given for $0.90\leq x\leq 0.99$ and $1.01\leq x\leq 10.00$ calculated with an accuracy to 10^{-7}. *S. Kulik* (Long Beach, Calif.)

11159:

Carlitz, L. A note on the Laguerre polynomials. Michigan Math. J. 7 (1960), 219–223.

The identity

$$\binom{m+n}{m}L^{(\alpha)}_{m+n}(x) = \sum_{r=0}^{\min(m,n)}\frac{(-1)^r}{r!}x^r L^{(\alpha+n+r)}_{m-r}(x)L^{(\alpha+r)}_{n-r}(x)$$

is established, where $L_n^{(\alpha)}(x)$ is the nth degree Laguerre polynomial of order α, which leads to other identities such as

$$\sum_{n=0}^\infty\binom{m+n}{m}L^{(\alpha-n)}_{m+n}(x)t^n = (1+t)^\alpha e^{-xt}L_m^{(\alpha)}[x(1+t)].$$

A. E. Danese (Schenectady, N.Y.)

11160:

Lord, R. D. Integrals of products of Laguerre polynomials. Math. Comp. 14 (1960), 375–376.

The coefficient

$$C_{rst} = \int_0^\infty e^{-x}L_r(x)L_s(x)L_t(x)\,dx$$

was evaluated by Watson [J. London Math. Soc. **13** (1938), 29–32]. The author varies Watson's technique, using the generating function of Laguerre polynomials where Watson used the equivalent contour integral, to obtain first a generating function for C_{rst}, then Watson's explicit formula, and then a recurrence relation and two more formulas. *A. Erdélyi* (Pasadena, Calif.)

11161:

Stein, Seymour. Addition theorems for spherical wave functions. Quart. Appl. Math. **19** (1961), 15–24.

The author first reviews the known transformation formulas for scalar spherical wave functions u_{mn} under rotations and translations of the coordinate system. He then develops the corresponding transformation formulas for the vector spherical wave functions

$$\mathbf{M}_{mn} = \nabla u_{mn}\times\mathbf{R},\ \mathbf{N}_{mn} = k^{-1}\nabla\times\mathbf{M}_{mn},$$

where $\mathbf{R}$ is the position vector.

A. Erdélyi (Pasadena, Calif.)

11162:

Bell, M. A note on Mathieu functions. Proc. Glasgow Math. Assoc. **3**, 132–134 (1957).

The difference of the periodic eigenvalues a_n, b_n of Mathieu's differential equation $d^2y/dx^2+(a-2q\cos 2z)y=0$ is examined and the leading term in the expansion of a_n-b_n in powers of q is found to be $q^n 2^{3-2n}[(n-1)!]^{-2}$.

J. Meixner (Aachen)

11163:

Tricomi, F. G. ★Fonctions hypergéométriques confluentes. Mémorial des Sciences Mathématiques, Fasc. CXL. Gauthier-Villars, Paris, 1960. iv+86 pp.

This is a clear, quite short, but comprehensive account of the confluent hypergeometric equation $zy''+(c-z)y'-ay=0$, based on the author's solutions $\Phi(a,\ c;\ z)$ and $\Psi(a,\ c;\ z)$ in preference to the Whittaker functions. The notation used is that of Erdélyi et al. in *Higher transcendental functions* [McGraw-Hill, New York, Vols. I, II, 1953, Vol. III, 1955] and in content the article reviewed is an expanded version of Chapters VI and IX of the former, which were contributed by the author.

Of the four chapters in this article, I and II introduce and describe the solutions Φ and Ψ, and the advantages gained by choosing these is made clear. Chapter III deals with asymptotic expansions and IV with particular cases and applications. Of this last, the section dealing with the incomplete gamma functions seems to the reviewer to be particularly valuable. *F. M. Arscott* (Battersea)

11164:

Rajagopal, A. K. On some of the classical orthogonal polynomials. Amer. Math. Monthly **67** (1960), 166–169.

C. K. Chatterjee [Bull. Calcutta Math. Soc. **49** (1957), 67–70; MR **20** #3308] proved that

$$y_n y_{n+1} = 1+\sum_0^n(2k+1)xy_k^2.$$

He also obtained analogues of Rodrigues' formula. The present note contains similar results for all the classical orthogonal polynomials except those of Gegenbauer and Jacobi (omitted because the results are cumbersome).

N. D. Kazarinoff (Ann Arbor)

ORDINARY DIFFERENTIAL EQUATIONS

See also 11074, 11162, 11205, 11206, 11286, B11513.

11165:

Redheffer, Raymond M. The Mycielski-Paszkowski diffusion problem. J. Math. Mech. **9** (1960), 607–621.

For a generalization of the diffusion problem considered by Mycielski and Paszkowski [Studia Math. **15** (1956), 188–200; MR **19**, 588] the author establishes that the involved probability functions $P_{ij}(x,\ y)$, $p_{ij}(x,\ y)$, $Q_{ij}(x,\ y)$, $q_{ij}(x,\ y)$ and $R_{ij}(x,\ y)$, $r_{ij}(x,\ y)$ ($i,\ j=1,\ \cdots,\ n$) of transmission, reflection and absorption satisfy a set of functional equations, and that the method used by the reviewer [J. Math. Mech. **8** (1959), 221–230; MR **22** #791] yields a set of matrix differential equations

$$\begin{aligned}
&P_y = P(K+\lambda Q), && p_y = (k+Q\lambda)p,\\
(*)\quad &Q_y = \Lambda+kQ+QK+Q\lambda Q, && q_y = P\lambda p,\\
&R_y = M+\mu Q+RK+R\lambda Q, && r_y = (R\lambda+\mu)p,
\end{aligned}$$

in the corresponding $n\times n$ matrices $P(x,\ y)$, $p(x,\ y)$, $Q(x,\ y)$, $q(x,\ y)$, $R(x,\ y)$, $r(x,\ y)$; the elements of the $n\times n$ coefficient matrices $\lambda(y)$, $k(y)$, $\mu(y)$, $\Lambda(y)$, $K(y)$, $M(y)$ are piecewise continuous functions on an interval I of consideration, and the $3n\times 3n$ matrix $S=S(x,\ y)$ defined by

$$S = \begin{Vmatrix} p & Q & 0 \\ q & P & 0 \\ r & R & 1 \end{Vmatrix}$$

reduces to the identity for $y=x$. Using an extension of the "star-product" of matrices introduced earlier by the author [ibid., 349–367; MR **22** #792], in the present paper it is shown that on $\{x, y \mid x \in I, y \in I, y > x\}$ the matrix $S=S(x, y)$ is dissipative in the sense that S has non-negative elements and $1 \geq \|S\| \equiv \sup \sum_{\alpha,\beta=1}^{3n} S_{\alpha\beta} v^\beta$ on $\{v \mid v \equiv (v^\beta)$ $(\beta=1, \cdots, 3n)$, $v^\beta \geq 0$, $\sum_{\beta=1}^{3n} v^\beta = 1\}$ if and only if the $3n \times 3n$ matrix

$$S_y(y, y) = \begin{Vmatrix} k(y) & \Lambda(y) & 0 \\ \lambda(y) & K(y) & 0 \\ \mu(y) & M(y) & 0 \end{Vmatrix}$$

is such that its off-diagonal elements are non-negative, and the sum of the elements in each column is non-positive for all $y \in I$. In particular, the result of the present paper settles a question raised by Mycielski and Paszkowski in the special case considered by them, for which the coefficient matrices of (*) are all constant.

W. T. Reid (Iowa City, Iowa)

11166:

Grunsky, Helmut. Ein konstruktiver Beweis für die Lösbarkeit der Differentialgleichung $y' = f(x, y)$ bei stetigem $f(x, y)$. Jber. Deutsch. Math. Verein. **63**, Abt. 1, 78–84 (1960).

Let $f(x, y)$ be continuous in a domain D of the (x, y)-plane and let $(x_0, y_0) \in D$. We consider the initial value problem of finding a solution $y = \varphi(x)$ of the differential equation $y' = f(x, y)$ such that $y_0 = \varphi(x_0)$. There exists at least one solution which tends in both directions to the boundary of D. If there is more than one solution, there exist two distinguished ones, namely, $\varphi_1(x)$ and $\varphi_2(x)$ such that each solution $\varphi(x)$ satisfies $\varphi_1(x) \leq \varphi(x) \leq \varphi_2(x)$. The author gives a construction for $\varphi_1(x)$ and $\varphi_2(x)$ which is a variant of Euler's polygonal method and is modelled on the Darboux construction for the upper and lower sums in the theory of definite integrals. The basic idea is as follows. The segment of the x-axis considered is subdivided into a set T of intervals; in each interval the side of the approximating polygon is given the least (largest) possible slope α such that on this side holds always $\alpha \geq f(x, y)$ $(\alpha \leq f(x, y))$. It is shown that with proper refinement of T these polygons converge monotonically to $\varphi_2(x)(\varphi_1(x))$.

M. Schiffer (Stanford, Calif.)

11167:

Wittich, Hans. Eindeutige Lösungen der Differentialgleichung $w' = R(z, w)$. Math. Z. **74** (1960), 278–288.

If $R(z, w)$ is rational in z and w, then, according to a previous result of the author, the equation (1) $w' = R(z, w)$ cannot have a solution which is single-valued in $R \leq |z| < \infty$, and has a transcendental singularity at $z = \infty$ unless (1) is a Riccati equation. In the present paper the nature of these single-valued solutions is studied in greater detail. If the Riccati equation is written in the normal form (2) $w' = Q(z) + w^2$, where $\lim_{r\to\infty} z^{-\nu} Q(z) = a \neq 0$, $|z| = r$, and if $T(r, w)$ denotes the Nevanlinna characteristic of w, the main result is the following: A solution of (2) which is single-valued in $R \leq |z| < \infty$ and satisfies (3) $\lim\inf_{r\to\infty} T(r, w)/\log r = \infty$, is of regular growth and has the order $\lambda(w, \infty) = 1 + \frac{1}{2}\nu$. For $\nu < -1$, there exists no single-valued solution with the property (3).

Z. Nehari (Pittsburgh, Pa.)

11168:

Liu, Yung-ching. Extension of a problem of type of centre on the torus. Advancement in Math. **4** (1958), 132–138. (Chinese. English summary)

This paper deals with differential equations of very special form, all of whose solutions except one are periodic on the torus.

S. S. Shu (Lafayette, Ind.)

11169:

Koepsel, Wellington W. Jump resonance in a third order nonlinear control system. J. Franklin Inst. **271** (1961), 292–303.

This article analyzes the third-order differential equation

$$A\frac{d^3c}{dt^3} + B\frac{d^2c}{dt^2} + C\frac{dc}{dt} + Dc + E^3 = F\cos(\omega t - \phi)$$

by means of solving for the first harmonic approximation using the Ritz-Galerkin method. The term labeled F is investigated and it is shown that jump resonance is determined by the magnitude of F and the phase angle ϕ between the applied forcing function and the approximated first harmonic solution. This is done by a graphical technique. The author does not have a careful explanation of the terms or the methods in the graphical technique.

R. Roy (Troy, N.Y.)

11170:

Cartwright, Mary L. Reduction of systems of linear differential equations to Jordan normal form. Ann. Mat. Pura Appl. (4) **51** (1960), 147–160.

The author examines various methods of finding a non-singular matrix P such that $PAP^{-1} = J$, where J is the Jordan normal form of A, with special reference to the problem of reducing the system of first-order differential equations with constant coefficients $\dot{x} = Ax$ to the form $\dot{y} = Jy$ where $y = Px$. In particular, for a differential equation of order n with constant coefficients $\sum_s b_s D^{n-s} x = 0$, where s ranges from 0 to n, and $D = d/dt$, the $n \times n$ matrix A has the typical form $B = (b_{ij})$, $b_{i,i+1} = 1$, $i = 1, \cdots, n-1$, $b_{ns} = -b_{n-s+1}$, $s = 1, \cdots, n$, $b_{ij} = 0$ otherwise, and devices are discussed to reduce B first to a canonical form J_0 analogous to J, where each diagonal block has the same form as B above. This form J_0 is denoted as the standard canonical form of B.

L. Cesari (Ann Arbor, Mich.)

11171:

Buslaev, V. S.; Faddeev, L. D. Formulas for traces for a singular Sturm-Liouville differential operator. Dokl. Akad. Nauk SSSR **132** (1960), 13–16 (Russian); translated as Soviet Math. Dokl. **1**, 451–454.

Consider $[Lu](x) = -u''(x) + q(x)u(x)$ over $0 < x < +\infty$, with the domain of the operator L in $L_2(0, \infty)$ being, say, all $u \in c_2(0, \infty) \cap L_2(0, \infty)$ which have $u(0^+) = 0$, where measurable q satisfies $\int_0^\infty x|q(x)|\alpha x < +\infty$ and also apparently (not stated by the authors but inferred from their conclusions) q is real-valued and of such a nature that L possesses a unique self-adjoint extension L in $L_2(0, \infty)$. The authors state that the resolvent R_λ of L, and ${}_0R_\lambda$ for

${}_0Lu = -u''$ similarly, have $R_\lambda - {}_0R_\lambda$ compact and of finite trace for complex λ not in the spectrum of L; they also give a formula for this trace. Assuming $q \in C_k(0, \infty)$, they also get additional formulae for the trace of certain order iterates depending upon k. Only barest hints at proofs are given, but reference is made to a preceding paper by the second author. *F. H. Brownell* (Seattle, Wash.)

11172:

Bollermann, Werner. Zur Einschliessung von Eigenwerten unter Verwendung des Maximum-Minimum-Prinzips. Z. Angew. Math. Mech. **40** (1960), 342–349. (English and Russian summaries)

The author shows how to obtain lower as well as upper bounds for the eigenvalues of a problem involving ordinary differential operators in terms of the eigenvalues and eigenvectors of a related problem. The method is a special case of that of A. Weinstein [see, e.g., S. H. Gould, *Variational methods for eigenvalue problems*, Univ. of Toronto Press, Toronto, 1957; MR **19**, 287]. Some simplifications are given, which are closely akin to methods proposed by W. Börsch-Supan [Math. Ann. **134** (1958), 453–457; MR **20** #1690] and the reviewer [Inst. for Fluid Dynam. & Appl. Math. Tech. Notes BN-41 (1954) and BN-183 (1959), Univ. of Md.].

H. F. Weinberger (Minneapolis, Minn.)

11173:

Bellman, Richard; Lehman, Sherman. Functional equations in the theory of dynamic programming. X. Resolvents, characteristic functions and values. Duke Math. J. **27** (1960), 55–69.

In the present paper the authors extend methods and results of earlier papers in which they applied the functional-equation technique of dynamic programming to the derivation of variational relations for kernels and Green's functions associated with certain problems in differential equations and integral equations. In general, the method of the present paper associates the solution of a Sturm-Liouville problem with a given function of a variational problem so constructed that the solution of the Sturm-Liouville problem maximizes the given function of the variational problem. The dynamic programming technique is then used to determine the maximizing function of the variational problem. Typical of the problems considered is the following: Given the Sturm-Liouville problem

$$(pu')' + (q(x) + \lambda r(x))u = v(x),\ a < x < 1,\ u(a) = 0,$$
$$u(1) + \alpha u'(1) = 0,$$

where p, q, r and v are continuous on $a \leq x \leq 1$. This problem is known to have a unique solution, when λ is not a characteristic value, of the form $u(x) = \int_a^1 R(x, y, \lambda, a)v(y)dy$. It is proved that the resolvent function $R(x, y, \lambda, a)$ satisfies the variational equation

$$\frac{\partial R}{\partial a}(x, y, \lambda, a) = p(a)\frac{\partial R}{\partial x}(a, y, \lambda, a)\frac{\partial R}{\partial y}(x, a, \lambda, a)$$

for all λ not equal to a characteristic value. Further, variational equations are derived for the characteristic values and functions.

Similar results are obtained for the matrix differential equation $(P(t)x')' + \lambda Q(t)x = 0$. The extension of the method to integral equation problems is briefly sketched. A number of typographical errors and notational vagaries provide a minor obstacle for the reader.

P. E. Guenther (Cleveland, Ohio)

11174:

Leont'ev, A. F. Sequences of linear aggregates of solutions $y(z, \lambda_j)$ of the ordinary differential equation $Dy = \lambda_j y$. Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, pp. 195–206. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

Let D be an ordinary linear differential operator of order s whose coefficients are analytic functions of the independent variable z, and let $y_k(z, \lambda)$, $k = 1, 2, \cdots, s$, be a set of linearly independent solutions of the differential equation $Dy = \lambda y$. The author selects a sequence $\lambda_1, \lambda_2, \cdots$ of values of λ and considers the sequence of functions

$$P_m(z) = \sum_{j=1}^{p_m} \sum_{k=1}^{s} a_{mj}^{(k)} y_k(z, \lambda_j) \quad (m = 1, 2, \cdots).$$

He gives a number of examples in which the P_m are elementary functions, certain of the classical orthogonal polynomials, Bessel functions, and the like. He discusses problems involving the P_m that are similar to problems involving sequences of analytic functions which he discussed in earlier papers [see, for instance, Amer. Math. Soc. Transl. (2) **10** (1958), 1–12; MR **20** #983].

A. Erdélyi (Pasadena, Calif.)

11175:

Ši, Čžun-cy. Asymptotic behavior of the eigenfunctions of Sturm-Liouville's problem calculated by using perturbation theory. Dokl. Akad. Nauk SSSR **131** (1960), 1279–1282 (Russian); translated as Soviet Math. Dokl. **1**, 442–445.

Considering $[Au](x) = -u''(x) + p(x)u(x)$, with $u \in C_2[0, \pi]$ satisfying $u(0) = 0 = u(\pi)$ the domain of A, and assuming $p^{(k)}(x)$ exists Lipschitz-wise over $0 \leq x \leq \pi$ for some integer $k \geq 1$ and that $p^{(j)}(0) = 0 = p^{(j)}(\pi)$ for all j, $1 \leq j \leq k$, the author easily computes the first few terms of the formal perturbation series (in the perturbation p) for the eigenfunctions of A. This computation shows that some assertedly well-known asymptotic formulae (as the eigenvalue parameter tends to $+\infty$) for these eigenfunctions validate the appropriate first few terms of the perturbation series in such an asymptotic sense.

F. H. Brownell (Seattle, Wash.)

11176:

Funato, M.; Maekawa, T. On the existence of subharmonics for Duffing's equation. Math. Japan. **5** (1958/59), 77–82.

The authors establish the existence of subharmonic solutions of (1) $x'' + \alpha x + \beta x^3 = \varepsilon \cos \omega t$, $\alpha > 0$, $\beta > 0$, ε small, of any order p where p is an integer less than $\omega\alpha^{-1/2}$. If $x_0(t)$ is the solution of $x'' + \alpha x + \beta x^3 = 0$ of least period $2\pi p/\omega$, the subharmonics have the form $x_0(t+\tau) + \varepsilon x_1(t) + o(\varepsilon)$. The case of a slightly damped equation (2) $x'' + cx' + \alpha x + \beta x^3 = \varepsilon \cos \omega t$ is also handled.

The authors use a theorem of the reviewer [Mem. Amer. Math. Soc. No. 31 (1959); MR **21** #5785] for existence. They give the details of their reasoning using Fourier series and appropriate results from the theory of elliptic functions. The proof when p is even is much more involved than the proof when p is odd.

W. S. Loud (Minneapolis, Minn.)

11177:

Hukuhara, Masuo. La direction de Julia au point singulier fixe d'une équation différentielle ordinaire du premier ordre. Japan J. Math. **29** (1959), 5–8.

L'auteur considère l'équation $x^{\sigma+1}dy/dx = P(x, y)/Q(x, y)$, où σ est un entier positif quelconque et P et Q sont deux polynômes en y sans facteurs communs, dont les coefficients sont holomorphes en $x=0$. Après avoir modifié un lemme de T. Kimura [Comment. Math. Univ. St. Paul **2** (1953), 23–28; MR **15**, 311], il démontre que la direction du demiaxe réel positif est celle de Julia pour une solution $\varphi(x)$, si $S_0 = \bigcap_{\alpha>0}(\bigcap_{r>0} S_{\alpha,r})$ ne se réduit pas à un seul point, où $S_{\alpha,r}$ désigne l'adhérence de l'ensemble des valeurs $\varphi(x)$ prises dans le domaine angulaire $|\arg x| < \alpha$, $0 < |x| < r$, et qu'alors les valeurs exceptionnelles que $\varphi(x)$ ne prend pas sont des racines de l'équation algébrique $P(0, y) = 0$.

T. Ura (Kobe)

11178:

Ghizzetti, Aldo. Comportamento asintotico degli integrali dell'equazione differenziale $x''+x+\varphi(x')=0$. Ann. Mat. Pura Appl. (4) **51** (1960), 167–202.

The function $\varphi(u)$ is assumed to be increasing, $\varphi'(u)$ continuous, $\varphi(0)=0$, $\varphi'(0)=2p>0$, and there exist positive constants σ, L, α such that $|\varphi'(u)-2p| \leqq (\alpha+1)L|u|^\alpha$ if $|u| \leqq \sigma$. By using successive approximations, the author obtains explicit asymptotic formulas for the solutions of $x''+x+\varphi(x')=0$ in a neighborhood of $x=x'=0$. These solutions are then compared with the solutions of $x''+2px'+x=0$.

J. K. Hale (Baltimore, Md.)

11179:

Sokolov, P. V. On the paper of A. V. Phakadze and A. A. Šestakov, "On the classification of the singular point of a first order differential equation not solved for the derivative". Mat. Sb. (N.S.) **53** (**95**) (1961), 541–543. (Russian)

There are many errors in notation and in argument in the paper by Phakadze and Šestakov [Mat. Sb. (N.S.) **49** (**91**) (1959), 3–12; MR **22** #1724]. These are listed and corrected by Sokolov.

C. S. Coleman (Claremont, Calif.)

11180:

Pontryagin, L. S.; Rodygin, L. V. Periodic solution of a system of ordinary differential equations with a small parameter in the terms containing derivatives. Dokl. Akad. Nauk SSSR **132** (1960), 537–540 (Russian); translated as Soviet Math. Dokl. **1**, 611–614.

The authors consider the system (written in vector form)

$$\varepsilon dx/dt = f(x, y), \quad dy/dt = g(x, y)$$

with f, $g \in C^3$ and $\varepsilon > 0$ small. The "system of rapid motions" (*) $dx/d\tau = f(x, y)$ (y a parameter) is assumed to have exactly one non-degenerate periodic solution $x^*(\tau, y)$ for every y, its period being $T(y)$. The authors then introduce the "mean system": $dy/dt = \bar{g}(y)$, where $\bar{g}(y) = (1/T(y)) \int_0^{T(y)} g(x^*(\tau, y), y)d\tau$, and assume that it has a non-degenerate equilibrium point y_0. The main theorem then states that for sufficiently small ε the original system has a unique periodic solution $\{x(t, \varepsilon), y(t, \varepsilon)\}$ near the closed curve $\{x^*(\tau, y_0), y_0\}$, satisfying the following "nearness" properties: the period of the solution is $\varepsilon(T(y_0)+O(\varepsilon))$; further,

$$\|\{x(t, \varepsilon), y(t, \varepsilon)\} - \{x^*(T(y_0)\varphi(t, \varepsilon), y_0), y_0\}\| = O(\varepsilon)$$

for an appropriate "phase" $\varphi(t, \varepsilon)$, which depends smoothly on t; and the multipliers of the variational system depend in the expected way on the multipliers of the variational system of (*) at the solution $x^*(\tau, y_0)$ and the proper values of the Jacobian $\partial\bar{g}/\partial y$ at $y = y_0$.

J. J. Schäffer (Montevideo)

11181:

Višik, M. I.; Lyusternik, L. A. Initial jump for non-linear differential equations containing a small parameter. Dokl. Akad. Nauk SSSR **132** (1960), 1242–1245 (Russian); translated as Soviet Math. Dokl. **1**, 749–752.

Continuing their research into the behavior of the solutions of singular perturbation problems near the boundaries of the domain of definition, the authors consider $\varepsilon y'' + \phi(x, y, y') = 0$ and $y|_{x=0} = y_0$, $y'|_{x=0} = C\varepsilon^{-\beta}$, $\beta > 0$, and give sufficient conditions for the existence of a bounded limit as $\varepsilon \to 0$; in this case they say that "an initial jump occurs". The condition on $\phi(x, y, y')$ is that for large $|y'|$, $\phi(x, y, y')$ should behave essentially like $|y'|^{1+\alpha}\psi(x, y)$, $\psi(x, y) \geqq a^2 > 0$. (See also same Dokl. **121** (1958), 778–781 [MR **20** #5918] by the same authors, and, for the analogous results for $\phi(x, y, y')$ linear in y', see W. Wasow, Comm. Pure Appl. Math. **9** (1956), 93–113 [MR **18**, 39].)

G. Latta (Stanford, Calif.)

11182:

Ura, Taro; Kimura, Ikuo. Sur le courant extérieur à une région invariante; théorème de Bendixson. Comment. Math. Univ. St. Paul. **8** (1960), 23–39.

This paper continues the work of Ura [Funkcial. Ekvac. **2** (1959), 143–200; MR **22** #1727] on certain one-parameter transformation groups acting on an open, relatively compact subset A of a topological space which is often the euclidean plane. In the plane, and for uniform curve families, the authors establish the following generalization of a theorem of I. Bendixson [Acta Math. **24** (1901), 1–88]. Let M be a closed invariant subset of A, M_0 one of the connected components of M and C_0 a connected component of the set $A - M_0$. Suppose that there exists a neighborhood U_0 of M_0 in $C_0 \cup M_0$ such that $U_0 - M_0$ contains no singular point. Exactly one of the following holds. (1) There exists a characteristic in C_0 which approaches M_0 as $t \to \infty$ or as $t \to -\infty$. (2) For each neighborhood U of M_0 such that $U \cap C_0$ is not simply connected, there exist an infinite number of closed characteristics in $U \cap C_0$ no one of which is homotopic to zero in $U \cap C_0$.

W. R. Utz (Columbia, Mo.)

11183:

Langer, Rudolph E. Turning points in linear asymptotic theory. Bol. Soc. Mat. Mexicana (2) **5** (1960), 1–12.

Consider the ordinary differential equation $D^n u + \lambda P^{(n-1)}(z, \lambda) D^{n-1} u + \cdots + \lambda^n P^{(0)}(z, \lambda) u = 0$, where $D = d/dz$ and λ is a large complex parameter, and assume that z varies in a compact complex domain and that $P^{(k)}(z, \lambda)$ is expandable in a power series in $1/\lambda$: $P^{(k)}(z, \lambda) = \sum_{\nu=0}^{\infty} p_\nu^{(k)}(z)/\lambda^\nu$. It is now well-known that the asymptotic solution forms of the differential equation, as a becomes infinite, depend upon the configuration of roots of the auxiliary equation $\theta^n + p_0^{(n-1)}(z)\theta^{n-1} + \cdots p_0^{(0)}(z) = 0$, and

that these solution forms undergo abrupt changes at a turning point z where one or several auxiliary roots coalesce. The author, in this expository article, presents in outline the turning point theory as developed mainly by himself and his students.

R. W. McKelvey (Boulder, Colo.)

11184:

McLeod, J. B. The determination of phase shift. Quart. J. Math. Oxford Ser. (2) **12** (1961), 17–32.

If $q(r)$ satisfies suitable conditions, that solution of the differential equation

$$\psi''+[\lambda-q(r)-l(l+1)r^{-2}]\psi = 0$$

which vanishes at $r=0$ is asymptotically, as $r\to\infty$, a multiple of $\cos[r\lambda^{1/2}-(l+1)\pi/2-\eta_l]$, where η_l is a constant known as the phase shift. Suppose that (i) $q(r)$ is continuously differentiable for $r>0$, $q(r)=O(r^{c-2})$ as $r\to 0$, $0<c\leqq 2$; (ii) $w(r)=\lambda-v(r)=\lambda-q(r)-l(l+1)r^{-2}$ has only one simple zero r_0, and has no other zero, for $r>0$; (iii) $q(r)\to 0$ as $r\to\infty$; and either q is absolutely integrable, or q is conditionally integrable and q' is absolutely integrable at ∞. Then the author proves

$$\eta_l = \int_{(l+1/2)\lambda^{-1/2}}^{\infty} [\lambda-(l+1/2)^2r^{-2}]^{1/2}\,dr - \int_{r_0}^{\infty} w^{1/2}(r)\,dr + \frac{1}{4}\left\{\pi - \int_{r_0}^{\infty} \frac{\sin 2\omega}{r^2 w(r)}\,d[r^2w(r)]\right\},$$

where $\omega=\tan^{-1}\{[w^{1/2}(r)r^{-1/2}\psi(r)]/[r^{-1/2}\psi(r)]'\}$ and ψ is that solution of the differential equation vanishing at $r=0$. The first two infinite integrals fail to converge separately, and it is understood that the integrands are combined for integration. If (ii) is modified, a second exact formula may be proved; in the second formula $(l+1/2)^2$ is replaced by $l(l+1)$. The first two terms in the expression for η_l are the customary approximation, the last term is the error term.

Under certain further assumptions the author gives estimates for the error term for small and large values of λ. An investigation of these estimates for large λ shows that in case $c\leqq 1$ in (i), the error term may be of the same order as the approximation, while in case $c>1$ the error term is always of a smaller order than the approximation.

A. Erdélyi (Pasadena, Calif.)

11185:

Yoshizawa, Taro. Stability and boundedness of systems. Arch. Rational Mech. Anal. **6**, 409–421 (1960).

Stability theory usually deals with the perturbation of a given fixed solution of a system (1) $\dot{x}=F(t,x)$; theory of boundedness refers to the boundedness of any solution. This paper considers similar concepts but relative to the deviation between any two solutions, according to the following definitions (where $x(t;\,x_0,\,t_0)$ denotes the solution through $(x_0,\,t_0)$): (i) the system is said to be equi-distance-bounded if given $\alpha>0$, $t_0\geqq 0$, there exists $\beta(t_0,\alpha)>0$ continuous in t_0 such that $\|x_0-\bar{x}_0\|\leqq\alpha$ and $t\geqq t_0$ imply $\|x(t;\,x_0,\,t_0)-x(t;\,\bar{x}_0,\,t_0)\|\leqq\beta$; (ii) if β in (i) does not depend on t_0, the system is said to be uniform-distance-bounded; (v) the system is equi-stable if given $\varepsilon>0$, $t_0\geqq 0$, there exists $\delta(t_0,\varepsilon)>0$, continuous in t_0, such that $\|x_0-\bar{x}_0\|<\delta$, $t\geqq t_0$ imply $\|x(t;\,x_0,\,t_0)-x(t;\,\bar{x}_0,\,t_0)\|<\varepsilon$; (vi) if δ in (v) is independent of t_0, the system is said to be uniform-stable; (vii) the system is quasi-equi-asymptotically-stable if given $\varepsilon>0$, $t_0\geqq 0$, there exist $\delta_0(t_0)$, $T(t_0,\,\varepsilon)>0$ such that $\|x_0-\bar{x}_0\|<\delta_0$, $t\geqq t_0+T$ imply $\|x(t;\,x_0,\,t_0)-x(t;\,\bar{x}_0,\,t_0)\|<\varepsilon$; (ix) if δ_0 and T in (vii) are independent of t_0 the system is quasi-uniform-asymptotically-stable; (x) if (ii) and (ix) are satisfied, the system is uniform-asymptotically-stable; other similar definitions are omitted here for the sake of brevity. The implications between these concepts are made clear by means of several lemmas and examples. Several theorems extend Lyapunov's second method to the cases considered, viz., Theorem 1: If system (2) $\dot{x}=F(t,x)$, $\dot{y}=F(t,y)$ admits a Lyapunov function $V(t,\,x,\,y)$ which satisfies a Lipschitz condition with respect to x, y, $V'\leqq 0$ and $b(\|x-y\|)\leqq V\leqq a(t,\,\|x-y\|)$, where $a(t,\,r)$ and $b(r)$ are positive for $r>0$, continuous, increasing, $a(t,r)\to 0$ as $r\to 0$, then (1) is equi-stable. Theorem 2: if $a(t,\,r)$ in Theorem 1 is independent of t, (1) is uniform-stable. Theorem 3: If a, b in Theorem 1 satisfy $b(r)\to\infty$ as $r\to\infty$, (1) is equi-distance-bounded. Theorem 4: If a in Theorem 3 is independent of t, (1) is uniform-distance-bounded. Theorem 8: Let F satisfy a Lipschitz condition in x; in order that (1) be uniform-asymptotically-stable it is necessary and sufficient that a Lyapunov function V exists which satisfies a Lipschitz condition in $(t,\,x,\,y)$, $b(\|x-y\|)\leqq V\leqq a(\|x-y\|)$, $V'\leqq -c(\|x-y\|)$, where $a(r)$, $b(r)$, $c(r)$ are continuous and positive for $r>0$, $b(r)\to\infty$ for $r\to\infty$ and $a(r)\to 0$ for $r\to 0$.

J. L. Massera (Montevideo)

11186:

Yoshizawa, Taro. Existence of a bounded solution and existence of a periodic solution of the differential equation of the second order. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **33** (1960/61), 301–308.

This paper contains a criterion for the existence of a bounded solution of a scalar equation $\ddot{x}=F(t,\,x,\,\dot{x})$, and two criteria for a vector system $\dot{x}=F(t,x)$ to have all its solutions existing in the future; the statements are too long to be reproduced here. As an application, the existence of periodic solutions of equations of the type $\ddot{x}+f(x,\dot{x})+g(t,x)=p(t)$, g, p periodic, is established under conditions which are also quite involved. An illustration of this is that $\ddot{x}+k\sin x=p(t)$ always has a periodic solution if $|p(t)|\leqq k$.

J. L. Massera (Montevideo)

11187:

Peyovitch, T. Quelques théorèmes élémentaires des intégrales généralisées et leurs applications. Boll. Un. Mat. Ital. (3) **15** (1960), 1–6.

Let $x\geqq x_0$. If ϕ is absolutely continuous, positive, increasing, and unbounded; f is integrable and $\int_x^\infty f(t)dt=O(1)$ as $x\to\infty$; $\psi(x,\,y)$ is continuous for $|y|<B$, $\psi(x,\,0)=0$, and ψ satisfies a Lipschitz condition in y with a Lipschitz constant $\lambda(x)$ which is integrable and such that $\int_x^\infty \lambda(t)dt=O(1)$ as $x\to\infty$; and if

$$(*)\qquad \begin{aligned} \varepsilon &= \sup_{x\geqq x_0} \phi(x)\int_x^\infty \frac{\lambda(t)}{\phi(t)}\,dt < 1,\\ &\sup_{x\geqq x_0} \left|\phi(x)\int_x^\infty \frac{f(t)}{\phi(t)}\,dt\right| \leqq (1-\varepsilon)B; \end{aligned}$$

then the author proves that the differential equation

$$\frac{dy}{dx} = k\frac{\phi'(x)}{\phi(x)}\,y+f(x)+\psi(x,\,y),$$

with $k=1$, possesses an asymptotically bounded solution. Under analogous conditions, the differential equation with $k=-1$ possesses a one-parameter family of asymptotically bounded solutions. {There is no explanation of the concept "generalized integrals", and the reviewer does not understand the conditions involving $O(1)$. If O is replaced by o, the conditions (*) follow from the others for sufficiently large x_0, the result is trivial, and the reference in the title to generalized integrals becomes mysterious.}

A. Erdélyi (Pasadena, Calif.)

11188:

Protasov, V. I. Analytic solutions of a linear differential equation with rapidly growing coefficients. Issledovaniya po sovremennym problemam teorii funkciĭ kompleksnogo peremennogo, pp. 226–233. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

L'auteur considère une équation (1) $\sum_{n=0}^{\infty} a_n y^{(n)}(x) = f(x)$ $(a_0 \neq 0)$ où les coefficients sont rapidement croissants dans le sens que $\lim\sup|a_n|^{1/n}=\infty$. Si $\log A_n$ est la suite régularisée convexe de $\log a_n$ (pour cette notion, redécouverte par l'auteur, voir S. Mandelbrojt, *Séries adhérentes, régularisation des suites, applications* [Gauthier-Villars, Paris, 1952; MR **14**, 542]), (1) admet une solution unique $y(x)=\sum_0^\infty (c_n/n!)x^n$ dès que $f(x)=\sum_0^\infty (b_n/n!)x^n$ avec $\lim\sup|A_n b_n|^{1/n}\leq 1$, et l'on a $\lim\sup|A_n c_n|^{1/n}\leq 1$.

J.-P. Kahane (Montpellier)

11189:

Pontryagin, L. S. Optimal regulation processes. Uspehi Mat. Nauk **14** (1959), no. 1 (85), 3–20. (Russian)

The author presents a theory of optimal control developed by him and his associates. The existence and uniqueness for optimal control for linear systems is obtained and the maximal principle is presented as a necessary condition for an optimal control for a nonlinear system.

Consider the real differential system

$$dx^i/dt = f^i(x^1, \cdots, x^n, u^1, \cdots, u^m) = f^i(x, u) \quad (i = 1, \cdots, n)$$

where $f(x, u)$ and $\partial f(x, u)/\partial x$ are continuous in $(x, u) \in R^n \times \Omega$, where Ω is a non-empty compact set in R^m. Given an initial point $x_0 \in R^n$ and a target point $x_1 \in R^n$, we seek a control $U=(u(t), t_0, t_1, x_0)$ which transfers x_0 to x_1. That is, $u(t)$ is a measurable function on some finite interval $t_0 \leq t \leq t_1$ and the corresponding solution $x(t)$ of $dx/dt=f(x, u(t))$ with $x(t_0)=x_0$ is defined on $t_0\leq t\leq t_1$ with $x(t_1)=x_1$. Let Δ be the set of all controls which transfer x_0 to x_1.

Let there be given a functional on controls $L(U)=\int_{t_0}^{t_1} f^0(x(t), u(t))dt$ where $f^0(x, u)$ and $\partial f^0(x, u)/\partial x$ are continuous in $R^n\times\Omega$. The control U is called (minimal) optimal in case $L(U)\leq L(U^*)$ for all controls $U^*\in\Delta$. In case $f^0\equiv 1$ this yields the time optimal problem.

Define the real function $K(\tilde\psi, \tilde x, u)$ in $R^{n+1}\times R^{n+1}\times\Omega$ by

$$K(\psi_0, \psi_1, \cdots, \psi_n, x^0, x^1, \cdots, x^n, u^1, \cdots, u^m) = \sum_{\alpha=0}^{n} \psi_\alpha f^\alpha(x^1, \cdots, x^n, u^1, \cdots, u^m)$$

and also define $N(\tilde\psi, \tilde x)$ on $R^{n+1}\times R^{n+1}$ by

$$N(\psi_0, \psi_1, \cdots, \psi_n, x^0, x^1, \cdots, x^n) = \max_{u\in\Omega} K(\tilde\psi, \tilde x, u).$$

For a given control U any solution $\tilde x(t)$, $\tilde\psi(t)$ of the system

$$(*)\qquad \begin{aligned} \frac{dx^i}{dt} &= \frac{\partial K}{\partial\psi_i}(\tilde\psi, \tilde x, u(t)) \quad (i=0,1,2,\cdots,n),\\ \frac{d\psi_i}{dt} &= -\frac{\partial K}{\partial x^i}(\tilde\psi, \tilde x, u(t)) \end{aligned}$$

(here $x^0(t)=\int_{t_0}^t f^0(x(s), u(s))ds$ and $\partial K/\partial x^0\equiv 0$) yields a solution $x(t)$ of $\dot x^i=f^i(x, u(t))$ for $i=1, 2, \cdots, n$. The control $U=(u(t), t_0, t_1, x_0)$ in Δ is called extremal in case: (1) there exists a nowhere-vanishing absolutely continuous vector $\tilde\psi(t)=(\psi_0(t), \psi_1(t), \cdots, \psi_n(t))$ on $t_0\leq t\leq t_1$ such that the response $x(t)$, and the above $x^0(t)$, together with $\tilde\psi(t)$ satisfy the differential system (*). (2) $K(\tilde\psi(t), \tilde x(t), u(t))=N(\tilde\psi(t), \tilde x(t))$ for almost all t on $t_0\leq t\leq t_1$. (3) $\psi_0(t_0)\leq 0$ and hence $\psi_0(t)=\psi_0(t_0)\leq 0$ and $N(\tilde\psi(t_0), \tilde x(t_0))=N(\tilde\psi(t), \tilde x(t))=0$ for all t on $t_0\leq t\leq t_1$. Theorem (maximal principle): An optimal control U in Δ is extremal.

The technique of the proof is to embed U in a family of controls and use the first variation of the control problem. In many instances the maximal principle shows that the optimal control U must have a graph always on the boundary of Ω.

The author then considers the linear case $\dot x=Ax+Bu$ for a constant $n\times n$ matrix A and constant $n\times m$ matrix B. Also he assumes that Ω is a convex polyhedron with the origin $x=0$ as an interior point. Then if A and B and the edges of Ω are not too specially related, the time optimal control is piecewise constant and lies always at the vertices of Ω. Also the optimal control is unique. If A is stable, the eigenvalues of A have negative real parts; then there exists an optimal control transferring a given initial point x_0 to the origin of R^n. For the case of a stable matrix A the existence and uniqueness of optimal control permits the synthesis of the control, that is, there exists $v(x)$ such that the solutions of $\dot x=Ax+Bv(x)$ are precisely the responses, for all x_0, to the optimal control which transfers x_0 to the origin.

L. Markus (Minneapolis, Minn.)

11190:

Pontryagin, L. S. Optimal processes of regulation. Proc. Internat. Congress Math. 1958, pp. 182–202. (Russian) Cambridge Univ. Press, New York, 1960.

The author presents an exposition of the theory of optimal control. The existence and uniqueness of the optimal are proved for linear differential systems and the general principle of the maximal is shown to be necessary for an optimal control for nonlinear systems. The results are the same as those described in #11189.

L. Markus (Minneapolis, Minn.)

11191:

Boltyanskiĭ, V. G.; Gamkrelidze, R. V.; Pontryagin, L. S. Theory of optimal processes. I. The maximum principle. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 3–42. (Russian)

This paper presents all the details of the proof of the maximal principle for optimal controls. The method follows the outline described earlier by these authors; cf. #11189.

L. Markus (Minneapolis, Minn.)

11192:

Gamkrelidze, R. V. Optimal control processes for

bounded phase coordinates. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 315–356. (Russian)

Consider the optimal control problem of steering the solution $x(t)$ of $dx^i/dt = f^i(x^1, \cdots, x^n, u^1, \cdots, u^m)$ $(i=1, \cdots, n)$ from a given initial point x_0 to a target point x_1 in a bounded closed region $B \subset R^n$. The controls $U=(u(t), t_0, t_1, x_0)$ are piecewise continuous, piecewise smooth functions $u(t)$ lying in a compact set $\Omega \subset R^m$ for finite time intervals $t_0 \leq t \leq t_1$. We assume that $f(x, u) \in C^1$ in an open neighborhood of $B \times \Omega$ and that the boundary of B is smooth. A functional on controls $L(v) = \int_{t_0}^{t_1} L(x(t), u(t))dt$ is prescribed for all controls $U \in \Delta$, those which steer x_0 to x_1 with response $x(t) \subset B$. The author finds necessary conditions for U to yield a minimal value for $L(U)$ among all controls of Δ.

If x_0, x_1, and the response $x(t)$ to the optimal control U, lie in the interior of B then the usual maximal principle is the desired necessary condition; cf. #11189.

The main theorem of the paper is a modification of the maximal principle valid for the case when the response $x(t)$ to the optimal control lies everywhere on the boundary of B, and in a certain general position. The statement of the theorem, which involves Lagrange multipliers, is too long to quote here.

In the later part of the paper the author considers the case where the response $x(t)$ for the optimal control lies partly in the interior of B and partly on the boundary of B with a finite number of junction points. At each junction point, where $x(t)$ changes from the interior of B to the boundary of B, certain jump conditions must be satisfied. *L. Markus* (Minneapolis, Minn.)

11193:

Rozonoèr, L. I. L. S. Pontryagin maximum principle in the theory of optimum systems. I, II, III. Avtomat. i Telemeh. **20** (1959), 1320–1334, 1441–1458, 1561–1578 (Russian. English summary); translated as Automat. Remote Control **20** (1960), 1288–1302, 1405–1421, 1517–1532.

Let x be an n-dimensional vector describing the state of a system governed by the differential equations $dx/dt = f(x, u, t)$, where u is an r-dimensional control vector whose choice is constrained by a system of inequalities $\phi_j(u) \geq 0$. In I, the author considers the problem of choosing u so as to minimize (or maximize) the linear function $\sum_{i=1}^n c_i x_i(T)$, where T is fixed and $x(T)$ is free. In II the same problem is considered for $x(T)$ constrained to lie in a closed convex set, both for T fixed and T free. The author shows that in these cases, the maximum principle stated by Pontryagin [#11189] holds for an optimal control. For T fixed and f of the form $A(t)x + \psi(u)$, where $A(t)$ is an $n \times n$ matrix and ψ is a function whose range is in n-space, it is shown that the maximum principle is also sufficient for optimality. {Reviewer's note: Both Pontryagin and the author overlook a device used by Valentine [*Contributions to the calculus of variations 1933–1937*, pp. 403–447, Univ. of Chicago Press, Chicago, Ill., 1937] whereby the problem with the constraints $\phi_j(u) \geq 0$ can be reduced to a problem without constraints to which the theory of the Bolza problem can be applied. In particular, the maximum principle then follows from the Weierstrass condition by appropriate translation; the boundary conditions for the "impulse function" (or multiplier) follow immediately from the transversality condition.}

In III the author discusses the relationship between the maximum principle and dynamic programming. A discrete variant of the control process with various terminal conditions is also taken up. A maximum principle that an optimal control satisfies is given for these problems. Several simple examples are given in all three parts. *L. D. Berkovitz* (Santa Monica, Calif.)

11194:

Groza, L. A. Asymptotic expansion of solutions of second order ordinary differential equations in Banach spaces. Dokl. Akad. Nauk SSSR **121** (1958), 963–966. (Russian)

Some aspects of the asymptotic theory for $\varepsilon \to +0$, of differential equations of the form (1) $y'' = F(x, y, y', \varepsilon)$, where x and ε are real, are extended to the case that y is an element of a Banach space. The results include the construction of a fundamental system with known asymptotic expansion in the linear homogeneous case, and a proof that the solution of (1) with y and y' prescribed at one point tends to the solution of the formal limiting equation, provided F is linear in y', the coefficient of y' being negative. The proofs, which are indicated rather sketchily, resemble procedures well-known in the case that the Banach space is a space of real or complex-valued functions with the maximum-modulus norm. *W. Wasow* (Madison, Wis.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 11102, 11123, 11332a–b, B11513.

11195:

Garabedian, P. R. Partial differential equations with more than two independent variables in the complex domain. J. Math. Mech. 9 (1960), 241–271.

The author studies certain properties of the solutions of linear partial differential equations of the second order with analytic coefficients by extending them to the complex domain. More specifically, among other things, he (1) develops a method of analytic continuation and contour integration in the complex domain which yields an especially simple derivation of the classical expression for the solution of the Cauchy problem; (2) sketches a theory of mixed boundary-value problems for the wave equation in several independent variables; and (3) derives a reflection rule for the solutions of an elliptic equation which vanish on a given analytic surface. The reviewer was unable to find the statement of a single theorem. *C. B. Morrey, Jr.* (Berkeley, Calif.)

11196:

Müller, Claus. Aspects of differential equations in mathematical physics. Partial differential equations and continuum mechanics, pp. 3–8. Univ. of Wisconsin Press, Madison, Wis., 1961.

A discursive account of the relationship between physics and mathematics, mostly exemplified by Laplace's equation. It is argued that they are mutually beneficial. *H. W. Lewis* (Madison, Wis.)

11197:

★Séminaire d'analyse dirigé par Pierre Lelong. 2e année: 1958/59. Faculté des Sciences de Paris. Secrétariat mathématique, Paris, 1959. iii+105 pp. (mimeographed)

The reports of this seminar are reviewed separately: #11406, 11257, 11284, 11139, 11128, 11256, 11412, 11225, 11210, 11264, 11224; the last report, by P. Malliavin, was listed as MR **21** #5853.

11198:

Miltzlaff, Gerhard. Über einen Satz, das allgemeine Integral einer nichtlinearen Differentialgleichung mit Hilfe von Bedingungsgleichungen zu ermitteln. Wiss. Z. Hochsch. Schwermaschinenbau Magdeburg **4** (1960), 75–77. (Russian and English summaries)

Consider a non-linear, first-order partial differential equation of the form

$$(*)\qquad f(\partial z/\partial x_1, \cdots, \partial z/\partial x_n) = 0,$$

the independent variables and the dependent variable z not explicitly entering f. In this paper, a general solution of (*), i.e., a solution depending on an arbitrary function of $n-1$ variables, is constructed without quadratures. {Readers are cautioned of a misleading typographical error. It seems to the reviewer, at least, that the definitions (2.6) of p_i must have been intended for just $i>1$ with $p_1=z$.}

A. Douglis (College Park, Md.)

11199:

Iacob, Caius. Remarques sur le problème de Dirichlet modifié. Com. Acad. R. P. Romîne **8** (1958), 1107–1111. (Romanian. Russian and French summaries)

On considère un domaine plan borné Ω limité par des courbes assez régulières (C_0 extérieure; C_i $(i=1, \cdots, p)$ intérieures). Si u_i est la mesure harmonique de C_i, on détermine les constantes $C_{i,j}$ par les conditions que $U_j=\sum_{k=1}^{p} C_{j,k}u_k$ $(j=1, \cdots, p)$ admette sur C_j le flux -2π (normale intérieure à Ω) et le flux 0 sur les autres courbes intérieures. Alors la forme quadratique $C_{i,j}x_ix_j$ est définie positive et $U_j>0$. Applications, extensions.

M. Brelot (Urbana, Ill.)

11200:

Morrey, Charles B., Jr. Second order elliptic equations in several variables and Hölder continuity. Math. Z. **72** (1959/60), 146–164.

Sia $H^1(G)$ il completamento dello spazio delle funzioni u continue con le derivate prime in G rispetto alla norma: $\|u\|_1=\|u\|_{L_2(G)}+\sum_{i=1}^{n} \|D_iu\|_{L_2(G)}$ $(D_i=\partial/\partial x_i)$ e sia $H_0{}^1(G)$ $(\subset H^1(G))$ la chiusura rispetto a $\|u\|_1$ di $\mathscr{D}(G)$ (funzioni a supporto compatto in G). Una funzione $u\in H^1(G)$ sia soluzione dell'equazione del secondo ordine di tipo ellittico data sotto la forma:

$$(1)\qquad \int_G\left\{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}D_ju+b_iu+e_i\right)D_iv + \left(\sum_{j=1}^{n} c_jD_ju+du+f\right)v\right\} dx$$

per ogni v a supporto compatto in G. I coefficienti a_{ij} sono supposti misurabili e limitati, tali che si abbia: $(1-h)|\xi|^2\leqq\sum_{i,j}^{1\cdots n} a_{ij}\xi_i\xi_j\leqq(1+h)|\xi|^2, x\in G, 0<h<1$ mentre: b_i, $c_i\in L_{2p}$, $d\in L_p$ con $p>n/2$; detta $B(x_0, r)$ la sfera di centro x_0 e raggio r, e_i, f soddisfano ipotesi del tipo seguente:

$$(2)\qquad \begin{aligned} &\sup_{r<a} (r/a)^{2-n-\lambda} \int_{B(x_0,r)} |f(x)|\, dx < +\infty;\\ &\sup_{r<a} (r/a)^{2-n-2\lambda} \int_{B(x_0,r)} |e_i(x)|^2\, dx < +\infty(0<\lambda<1).\end{aligned}$$

(i) Supponendo che $u\in H_0{}^1(G)$ (la relazione (1) si riduce ad un problema di Dirichlet omogeneo) il problema di autovalori legato al problema considerato possiede uno spettro discreto di autovalori e per esso vale il teorema dell'alternativa. (ii) Supponendo che $u\in L_2(G)$, $u\in H^1(G')$ con $\bar{G}'\subset G$, sotto alcune limitazioni per le (2), una funzione che soddisfa la (1) è hölderiana nell'interno di G. Questo risultato generalizza precedenti risultati di Nash [Amer. J. Math. **80** (1958), 931–954; MR **20** #6592] e di De Giorgi [Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43; MR **20** #172] relativi al caso in cui: $b_i=e_i=c_i=d=f=0$. Esso si avvale in modo essenziale dei risultati di De Giorgi. (iii) G soddisfa la condizione $S^*(\alpha, a)$ se esiste una costante $\alpha>0$ tale che: mis$\{B(x_0, r)-G\}\geqq\alpha$ mis$B(x_0, r)$ per ogni $r\leqq a$, $x_0\notin G$: Verificata l'ipotesi $S^*(\alpha, a)$ per G, delle limitazioni del tipo (2) per f ed e_i, allora la soluzione u del problema di Dirichlet per l'equazione (1) $(u\in H_0{}^1(G))$ è hölderiana in $\bar{G}$. Questo risultato generalizza parzialmente un precedente risultato del recensore [Ann. Scuola Norm. Sup. Pisa (3) **12** (1958), 223–245] dedicato allo studio della limitatezza della soluzione di problemi al contorno anche diversi da quello di Dirichlet.

G. Stampacchia (Pisa)

11201:

Beckert, Herbert. Eine bemerkenswerte Eigenschaft der Lösungen des Dirichletschen Problems bei linearen elliptischen Differentialgleichungen. Math. Ann. **139**, 255–264 (1960).

Sia F un operatore differenziale lineare del secondo ordine di tipo ellittico, i cui coefficienti siano sufficientemente "regolari" in un dominio E, esso pure "regolare", dello spazio R^n e tali che il problema di Dirichlet $Fu=f$ in E, $u=\varphi$ in S (S frontiera di E, φ continua su S, f hölderiana in $E+S$) ammetta una e una sola soluzione regolare e questa sia esprimibile con la classica formula

$$(1)\qquad k_nu(x) = \int_S \varphi(y) \frac{\partial G(x, y)}{\partial \nu_y}\, d\sigma_y - \int_E G(x, y)f(y)\, dy$$

(ν_y "conormale" in y a S). Sia poi U una fissata porzione di S e Γ una varietà s-dimensionale $1\leqq s\leqq n-1$ interna a E e che non sconnette E. L'A. dimostra che, tenuti fissi f in E e φ su $S-U$, al variare di φ solo su U, la restrizione di u su Γ percorre una varietà M densa nello spazio H_Γ delle funzioni di quadrato sommabile su Γ. La dimostrazione parte dalla formula di rappresentazione (1) e sfrutta in modo essenziale il teorema di unicità per il problema di Cauchy per l'operatore F. In modo analogo si dimostra che M é densa anche nello spazio delle funzioni di quadrato sommabile su Γ insieme alle derivate prime (rispetto ai parametri locali di Γ). Tra le possibili estensioni dei risultati suddetti alle equazioni ellittiche d'ordine superiore viene sviluppata quella relativa al problema di Dirichlet per l'equazione $\Delta\Delta u=f$ (Δ laplaciano).

E. Magenes (Pavia)

11202:

Slobodyanskiĭ, M. G. Construction of the principal part of the Green function for a second-order elliptic operator. Uspehi Mat. Nauk **13** (1958), no. 6 (84), 161–166. (Russian)

Consider the second-order linear elliptic partial differential equation

$$Lu=\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2u}{\partial x_i\,\partial x_j}+\sum_{i=1}^{n}b_i\frac{\partial u}{\partial x_i}+cu=f.$$

In connection with some questions of bounding the unknown function $u(x)=u(x_1,\cdots,x_n)$ and its partial derivatives [Dokl. Akad. Nauk SSSR **89** (1953), 221–224; MR **17**, 195; Trudy Moskov. Ènerg. Inst. **17** (1955), 122–141] the author was led to the determination of a fundamental solution $v(x,y)$ of the differential operator L such that $L[v(x,y)]=O(r^{-(n-1)+m})$, where $r^2=\sum_{i=1}^{n}(x_i-y_i)^2$ and $m>0$. The construction is carried out here.

J. B. Diaz (College Park, Md.)

11203:

Myrberg, Lauri. Über subelliptische Funktionen. Ann. Acad. Sci. Fenn. Ser. A I No. 290 (1960), 9 pp.

The author considers subfunctions for the equation $\Delta u=cu$ on a Riemann surface F, where c is a smooth non-negative density. Let ω be the largest non-negative solution of $\Delta\omega=c\omega$ on F which is bounded by 1. The author proves that if for a subfunction u we have $u\leqq\omega$ everywhere on F then either $u\equiv\omega$, or $u<\omega$ everywhere. Several further results are proved in analogy with subharmonic functions. The influence of the density c on the class of subfunctions is also considered.

H. L. Royden (Stanford, Calif.)

11204:

Tutschke, Wolfgang. Bemerkungen über das Verhalten von elliptischen Differential-Operatoren zweiter Ordnung bei konformer Abbildung. Math. Nachr. **22** (1960), 204–224.

The author finds conditions under which a second-order elliptic differential operator in two variables with smooth coefficients can be transformed by a conformal mapping of the independent variables into a differential operator whose second-order terms have constant coefficients. The author shows that, in addition to having the ratio of the eigenvalues of the matrix of the form constant, one must also have certain integrability conditions satisfied by the coefficients of the form.

H. L. Royden (Stanford, Calif.)

11205:

Diaz, J. B.; Walter, W. L. On uniqueness theorems for ordinary differential equations and for partial differential equations of hyperbolic type. Trans. Amer. Math. Soc. **96** (1960), 90–100.

The authors give proofs for the standard uniqueness theorems for the scalar initial-value problem $y'=f(x,y)$, $y(x_0)=y_0$ under the conditions that either f is subject to a uniform Lipschitz condition with respect to y or that f satisfies a Nagumo condition. The authors prefer their arguments which depend only on the Lagrange mean-value theorem, rather than on the more complicated mean-value theorem of integral calculus. They use similar arguments to obtain uniqueness theorems for $u_{xy}=F(x,y,u,u_x,u_y)$, $u(x,0)=\sigma(x)$, $u(0,y)=\tau(y)$ for $0\leqq x\leqq a$, $0\leqq y\leqq b$ under the conditions that F is continuous and satisfies an inequality

$$xy|F(x,y,u_1,p_1,q_1)-F(x,y,u_2,p_2,q_2)|$$
$$\leqq\alpha(x,y)|u_1-u_2|+\beta(x,y)|p_1-p_2|+\gamma(x,y)|q_1-q_2|$$

for $xy\neq0$ and corresponding inequalities for $x\neq0$, $y=0$ and for $x=0$, $y\neq0$, where the functions α, β, γ are continuous and satisfy $\alpha+\beta+\gamma=1$. A uniqueness theorem of this type for systems, and other theorems for $u_{xy}=F$ related to uniqueness theorems for $y'=f$, were obtained at the same time by J. P. Shanahan [Pacific J. Math. **10** (1960), 677–688; MR **22** #6924]; more general theorems have since been obtained by W. Walter [Math. Z. **74** (1960), 191–208; MR **22** #8206].

P. Hartman (Baltimore, Md.)

11206:

Diaz, J. B. On existence, uniqueness, and numerical evaluation of solutions of ordinary and hyperbolic differential equations. Ann. Mat. Pura Appl. (4) **52** (1960), 163–181.

The essentially new result is the following (Theorem 6): If the real-valued function $f(x,y,z)$ is defined for all (x,y,z) such that $x_0<x<x_0+a$, $y_0<y<y_0+b$, $-\infty<z<+\infty$, where x_0, y_0, a, b, are real numbers, with $a>0$, $b>0$, and f satisfies a Lipschitz condition in the argument z, and if the real-valued function $\sigma(x)$ is defined for all x such that $x_0\leqq x<x_0+a$ and the real-valued function $\tau(y)$ is defined for all y such that $y_0\leqq y<y_0+b$, then there is at most one real-valued function $u(x,y)$ defined and continuous on the (partly open) rectangle $x_0\leqq x<x_0+a$, $y_0\leqq y<y_0+b$, which has a finite, continuous partial derivative u_x on the (partly open) rectangle $x_0<x<x_0+a$, $y_0\leqq y<y_0+b$, has a finite mixed partial derivative u_{xy} on the (open) rectangle $x_0<x<x_0+a$, $y_0<y<y_0+b$ and satisfies the partial differential equation $u_{xy}(x,y)=f(x,y,u(x,y))$ on the (open) rectangle $x_0<x<x_0+a$, $y_0<y<y_0+b$.

The author points out that the lack of symmetry of the hypotheses with respect to x and y is due to the fact that the mixed derivative u_{xy} is taken first with respect to x and then with respect to y; nothing is assumed about the existence of the partial derivative u_{yx}, or even of the first derivative u_y.

The above result corresponds to a uniqueness theorem (Theorem 5) for the ordinary differential equation $y'=f(x,y)$ due to the author and W. Walter and already published elsewhere [#11205].

R. Conti (Florence)

11207:

Owens, O. G. An ultrahyperbolic equation with an integral condition. Amer. J. Math. **82** (1960), 799–811.

The author recently proved [Trans. Amer. Math. Soc. **88** (1958), 388–399; MR **20** #1076] that $(\partial^2/\partial x^2+\partial^2/\partial y^2+\lambda^2)v=0$,

$$(1)\qquad \int_0^\infty v(r\cos\theta,r\sin\theta)\,dr=f(\theta)$$

has a unique solution. Here this is applied to devise a well-posed problem for $\sum_{i=1}^{2}(\partial^2/\partial x_i^2-\partial^2/\partial t_i^2)u=0$. In the product of a finite x-region and a whole t-plane $u=0$ is prescribed along the boundary together with a condition like (1) on interior planes.

P. Ungar (New York)

11208:

Kapilevič, M. B. Transformation operators connected with singular Goursat problems. Dokl. Akad. Nauk SSSR **130** (1960), 487–490 (Russian); translated as Soviet Math. Dokl. **1**, 57–60.

Let $L_p{}^q(\delta)$ $(p, q=0, 1, \cdots, \infty)$ be the set of functions f defined on $\delta=\{y|0\leq y\leq y_0\}$, continuously differentiable p times on δ and satisfying the conditions $f(0)=f'(0)=\cdots=f^{(q)}(0)=0$. The first singular Goursat problem is the problem of finding in the domain D: $\{(x, y)|0\leq x\leq x_0, 0\leq y\leq y_0\}$ those solutions $z(x, y, b)$ of the equation $xz_{xy}+az_x+bz_y=0$ $(a>0, b>0)$ which are continuous in D together with their derivatives of order p, and which take the values $z(0, y)=f(y)$, $z(x, 0)=0$, $f\in L_p{}^q(\delta)$. A series of theorems giving the dependence of z on the exponent b of the singular characteristic $x=0$ are given. The maximum principle is shown to hold for the singular Goursat problem. *N. D. Kazarinoff* (Ann Arbor, Mich.)

11209:

Pini, Bruno. Sulle equazioni lineari del quarto ordine in due variabili con caratteristiche coincidenti. II. Atti Sem. Mat. Fis. Univ. Modena **9** (1959/60), 59–113.

In this paper boundary-value problems similar to those of (I) [same Atti **8** (1958/59), 130–166; MR **21** #7359], namely, $\mathscr{L}[u]=f$ in R: $[0<x<1, 0<y<h]$, u given on $\mathscr{F}R$, $\partial u/\partial y$ given on $y=0$, and either $\partial u/\partial x$ or $\partial^2u/\partial x^2$ given on the vertical sides of $\bar{R}$, are examined for the operator

$$\mathscr{L}[u]=\mathscr{L}_0[u]+\sum_{\substack{0\leq h+k\leq 3\\ k<3}}a_{hk}(x, y)(\partial^{h+k}u/\partial x^h\partial y^k)=f(x, y),$$

where $\mathscr{L}_0[u]=\partial^4u/\partial x^4-\partial^3u/\partial y^3$. Essentially the same procedure is followed as in (I). A fundamental solution is constructed for $\mathscr{L}_0[u]=0$, from which a solution for $\mathscr{L}_0[u]=f$ is obtained by the method of domain potentials. The groundwork is laid for the translation of each of the two boundary-value problems into a system of integral equations, but the process is not carried to completion. The respective Green's functions are constructed by the use of Fourier transforms. A method used previously by the author [Rend. Sem. Fac. Sci. Univ. Cagliari **26** (1956), 30–57; MR **18**, 314] is used to prove uniqueness of the solution of the second problem and is obviously applicable to the first also. *R. N. Goss* (San Diego, Calif.)

11210:

Huet, Denise. Perturbation singulière d'opérateurs elliptiques. Séminaire P. Lelong, 1958/59, exp. 13, 7 pp. Faculté des Sciences de Paris, 1959.

Quelques remarques autour de notes de l'A. aux C. R. Acad. Sci. Paris **244** (1957), 1438–1440; **246** (1958), 2096–2098; **247** (1958), 2273–2276; **248** (1959), 58–60 [MR **19**, 421; **20** #2540; **21** #758, 2831]. *J. L. Lions* (Nancy)

11211:

Avantaggiati, Antonio. Su un problema al contorno per un sistema ellittico di equazioni lineari alle derivate parziali del prim'ordine in tre variabili. Ricerche Mat. **9** (1960), 177–202.

In this paper, the author supplies the detailed proofs of the results which he announced in a note with the same title in the Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 332–335 [MR **22** #8219]. Some additional refinements are included. *C. B. Morrey, Jr.* (Berkeley, Calif.)

11212:

Hellwig, Günter. Über Systeme hyperbolischer Differentialgleichungen erster Ordnung. III. Math. Z. **68** (1958), 325–337.

[Parts I and II (with W. Haack): same Z. **53** (1950), 244–266, 340–356; MR **12**, 614.] In the first part of this paper linear, semilinear and quasilinear hyperbolic systems of first-order equations in two variables are reduced to normal (diagonal) form. In the quasilinear case such a reduction is possible for more than two unknown functions only if certain compatibility conditions are satisfied. Then characteristic initial-value problems are investigated for linear systems in normal form; these consist of prescribing the values of the characteristic variables on various characteristic curves through a given point. Among the possible characteristic initial-value problems a certain class is distinguished which occurs in the construction of the Riemann function, following Holmgren and Rellich.

Throughout the paper good use is made of the language of exterior differential forms. *P. Lax* (New York)

11213:

Gal'pern, S. A. Lacunae of non-hyperbolic equations. Dokl. Akad. Nauk SSSR **132** (1960), 990–993 (Russian); translated as Soviet Math. Dokl. **1**, 680–683.

Let $Q(\partial/\partial t, \partial/\partial x_1, \cdots, \partial/\partial x_n)$ be a differential operator with constant coefficients in which only terms of order l occur, and the highest power of $\partial/\partial t$ is the mth. Assume the characteristic normal cone has $\frac{1}{2}m$ disjoint sheets of multiplicity 1, and that these have no generator parallel to the t-axis or the $t=\text{const}$ plane. The author gives a formula for a fundamental solution $QK(x, t)=0$, $\partial^iK(x, 0)/\partial t^i=\delta_{i\,m-1}\delta(x)$ $(i=0, 1, \cdots, m-1)$ and notes that it vanishes in a cone containing the t-axis. Q is a limit of hyperbolic operators with lacunae. *P. Ungar* (New York)

11214:

Garabedian, P. R. Stability of Cauchy's problem in space for analytic systems of arbitrary type. J. Math. Mech. **9** (1960), 905–914.

The author discusses the Cauchy problem for an arbitrary analytic system of equations in the complex domain by first reducing the system to one of the first order with a distinguished variable t in the usual way. The Cauchy problem for this system is then handled by introducing an appropriate symmetric hyperbolic system to determine the real and imaginary parts of the unknown functions in terms of the $2n+1$ variables $t, x_1, \cdots, x_n, y_1, \cdots, y_n$, where $t, z_1, \cdots, z_n$ are the independent variables in the first-order system. The author remarks that the Cauchy problem thus obtained is stable, and also that analyticity with respect to the variable t is not required. The author emphasizes that the method is independent of the type of the system and illustrates this with a description of a stable finite-difference scheme applicable to the flow equations in the trans-sonic region. *C. B. Morrey, Jr.* (Berkeley, Calif.)

11215:

Riesz, Marcel. A geometric solution of the wave equation in space-time of even dimension. Comm. Pure Appl. Math. **13** (1960), 329–351.

To find $u(0)$, a coordinate system based on the characteristic cone through the point 0 is used. Integrating the suitably transformed equation over the characteristic cone everything drops out except $u(0)$ and an integral over the initial surface. The geometric significance of the objects appearing in the argument is elucidated.

P. Ungar (New York)

11216:

Miranker, W. L. A well posed problem for the backward heat equation. Proc. Amer. Math. Soc. **12** (1961), 243–247.

This paper treats the Cauchy problem for the backward heat equation $u_t = -u_{xx}$, $u(0, x) = f(x)$, $t > 0$, where f belongs to the subspace of L_2 functions whose Fourier transforms have compact support. It is shown that in this subspace the problem is well-posed in the sense of Hadamard.

J. Elliott (New York)

11217a:

Shirota, Taira. A unique continuation theorem of a parabolic differential equation. Proc. Japan Acad. **35** (1959), 455–460.

11217b:

Shirota, Taira. A remark on my paper "A unique continuation theorem of a parabolic differential equation". Proc. Japan Acad. **36** (1960), 133–135.

These two papers concern the extension of a unique continuation theorem for elliptic differential equations to parabolic equations. The second paper is a simplification of the first. Consider real solutions of the inequality

$$|u_t - \sum a_{ij}(t, x)u_{x_ix_j}| \leq M\{\sum |u_{x_i}| + |u|\}$$

in a convex subset G of $\{-\infty < t < \infty;\ -\infty < x_i < \infty,\ i = 1, \cdots, n\}$. Let $C: \{(t, x_i(t)) | t \in [a, b],\ x_i(t) \in C^1[a, b]\}$ be a curve in G. If $|u|$, $|u_{x_i}|$, $|u_{x_ix_j}|$ vanish as x approaches C faster than any power of $|x - x(t)|$, then u vanishes identically in the horizontal component $G \wedge \{(t, x) | t \in [a, b]\}$. The proof is based generally on the methods of Cordes and Heinz.

J. Douglas, Jr. (Houston, Tex.)

11218:

Agranovič, M. S. Existence of solutions of partial differential equations with constant coefficients in certain classes of functions. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 3, 3–13. (Russian)

Let $P = P(\partial/\partial x_1, \cdots, \partial/\partial x_n)$ be a differential operator with constant coefficients. Let $S_{\alpha,A}$ and $W_{M,a}$ be the spaces defined in Gel'fand and Šilov's books [*Prostranstva osnovnyh i obobščennyh funkciĭ* (Chapter IV); *Nekotorye voprosy teorii differencial'nyh uravneniĭ* (Chapter I); Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958; MR **21** #5142a, b]. Let $0 \leq \alpha \leq 1$; assume that for any $\varepsilon > 0$ there exists a $C > 0$ such that $M_k(\varepsilon t) \geq C\mu_k(t)$ for all sufficiently large t, where M_k and μ_k are the functions which occur in the definition of $W_{M,a}$. The author proves that $PW'_{M,a} = W'_{M,a}$ and $PS'_{\alpha,A} = S'_{\alpha,A}$.

S. Łojasiewicz (Kraków)

11219:

Dikopolov, G. V.; Šilov, G. E. Well-posed boundary problems for partial differential equations on a half-space. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 369–380. (Russian)

Let $\mathfrak{H}$ be the space of distributions on $\mathbf{R}^n$ generated by all derivatives of functions in L^2, with a natural topology. The Fourier transform H of $\mathfrak{H}$ is a space of functions. Consider the equation with constant coefficients: $\partial^m u/\partial t^m = \sum_0^{m-1} P_k(i\partial/\partial x)\partial^k u/\partial t^k$, P_k being polynomials. Let G_j denote the set: $\operatorname{Re} \lambda_j(\sigma) \leq 0$, $j = 1, \cdots, m$, where $\lambda_1(\sigma), \cdots, \lambda_m(\sigma)$ are the roots of $\sum_0^{m-1} P_k(\sigma)\lambda^k - \lambda^m = 0$, $\operatorname{Re} \lambda_1(\sigma) \leq \cdots \leq \operatorname{Re} \lambda_m(\sigma)$. In the class of solutions u such that $u(\cdot, t) \in \mathfrak{H}$ and $u(\cdot, t)/t^q$ is bounded in $\mathfrak{H}$ for some q if $0 \leq t < \infty$, the following boundary problem is considered: the Fourier transform of $\partial^j u/\partial t^j(\cdot, 0)$ coincides a.e. on G'_{j-1} with a given function from H, $j = 0, \cdots, m-1$, where $G_1' \supset \cdots \supset G_m'$. The problem is well-posed if and only if $(G_j' - G_j) \cup (G_j - G_j')$ is a set of measure zero, $j = 1, \cdots, m$.

There is also a necessary and sufficient condition in terms of Fourier transforms of Cauchy data, for existence and uniqueness (in the class considered above) of the solution of the Cauchy problem (on $t = 0$) for the system $\partial u_j/\partial t = \sum_{k=1}^m P_{jk}(i\partial/\partial x)u_k$, $j = 1 \cdots, m$.

The question of regularity of solutions is considered.

S. Łojasiewicz (Kraków)

11220:

Palamodov, V. P. Well-posed boundary problems for partial differential equations on a half-space. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 381–386. (Russian)

This is a continuation of some ideas contained in the paper reviewed above [#11219]. The system $\partial u_j/\partial t = \sum_{k=1}^m P_{jk}(i\partial/\partial x)u_k$, $j = 1, \cdots, m$ is considered (the same class of solutions as above). Let F_0, F_j $(j = 1, \cdots, m-1)$, F_m denote the sets: $0 < \operatorname{Re} \lambda_1(\sigma)$, $\operatorname{Re} \lambda_j(\sigma) \leq 0 < \operatorname{Re} \lambda_{j+1}(\sigma)$, $\operatorname{Re} \lambda_m(\sigma) \leq 0$, where $\lambda_j(\sigma)$ are the characteristic roots of the matrix $\|P_{jk}(\sigma)\|$, $\operatorname{Re} \lambda_1(\sigma) \leq \cdots \leq \operatorname{Re} \lambda_m(\sigma)$. Let v_j be the Fourier transform of $u_j(\cdot, 0)$. The following boundary problem is considered: $v_1 = \cdots = v_m = 0$ on F_0, and $\sum_{j=1}^m c_{kj}{}^{(\rho)}(\sigma)v_j(\sigma) = \varphi_k{}^{(\rho)}(\sigma)$, $k = 1, \cdots, \rho$, on F_ρ, $\rho = 1, \cdots, m$. A necessary and sufficient condition (in terms of $c_{kj}{}^{(\rho)}$, $\varphi_k{}^{(\rho)}$) is given for existence and uniqueness of a solution.

S. Łojasiewicz (Kraków)

11221:

Tong, Kwong-Chong. Uniqueness theorem for Chaplygin's problem. I, II. Acta Math. Sinica **6** (1956), 242–262. (Chinese. English summary)

Both papers deal with the uniqueness theorem of the boundary-value problem of Chaplygin's equation. The first is the refinement of Morewetz' theorem [Comm. Pure Appl. Math. **7** (1954), 697–703; MR **16**, 484] on Frankl's problem. The second is the improvement of Frankl's theorem [Izv. Akad. Nauk SSSR **9** (1945), 121–143; MR **7**, 496] on Tricomi's problem.

S. S. Shu (Lafayette, Ind.)

11222:

Il'in, A. M.; Oleĭnik, O. A. Asymptotic behavior of

solutions of the Cauchy problem for some quasi-linear equations for large values of the time. Mat. Sb. (N.S.) 51 (93) (1960), 191–216. (Russian)

Proofs of results announced earlier [Dokl. Akad. Nauk SSSR **120** (1958), 25–28; MR **21** #207].

L. Gårding (Lund)

11223:

Drăgilă, Pavel. Les caractéristiques des équations aux dérivées partielles linéaires. C. R. Acad. Sci. Paris **251** (1960), 1154–1156.

The author studies the characteristics of linear partial differential equations that are invariant under change of independent variables. It is shown that a hyperbolic system is characterized by three systems of characteristics but that a parabolic system has only one system of characteristics.

D. E. Spencer (Storrs, Conn.)

11224:

Peetre, Jaak. Comparaison d'opérateurs différentiels. Séminaire P. Lelong, 1958/59, exp. 16, 7 pp. Faculté des Sciences de Paris, 1959.

L'auteur donne dans cet exposé une esquisse de sa démonstration pour les résultats énoncés dans la note antérieure [C. R. Acad. Sci. Paris **248** (1959), 1102–1103; MR **21** #773].

S. Mizohata (New York)

11225:

Zerner, Martin. Équations aux dérivées partielles dont les solutions sont indéfiniment dérivables (hypoellipticité). Séminaire P. Lelong, 1958/59, exp. 12, 8 pp. Faculté des Sciences de Paris, 1959.

Nous savons que Hörmander a donné une caractérisation des opérateurs hypoelliptiques à coefficients constants [Acta Math. **94** (1955), 161–248; MR **17**, 853]. L'auteur montre ici, en suivant la méthode introduite par Malgrange [Bull. Soc. Math. France **85** (1957), 283–306; MR **21** #5063], un raisonnement assez simple pour arriver aux résultats de Hörmander. Cet exposé pourra servir grandement d'introduction à l'étude des opérateurs hypoelliptiques.

S. Mizohata (New York)

POTENTIAL THEORY

See also 11100, 11132, 11145, 11146, 11147, 11148, 11199, 11209, B11416, B11431, B11432.

11226:

Bouligand, Georges. La première phase de l'évolution ensembliste du problème de Dirichlet. Bull. Sci. Math. (2) **84** (1960), 111–115.

This brief historical article is concerned mainly with the contributions to the subject of the title made by the late Florin Vasilesco.

W. C. Fox (New Orleans, La.)

11227:

Wolf, František. On majorants of subharmonic functions. Bull. Soc. Math. Belg. **11** (1959), 22–26.

This paper is a report of a lecture given by the author at the Institut des Hautes Études de Belgique on January 19, 1960. It deals with problems related to the Phragmén-Lindelöf principle and conditions on subharmonic functions in certain domains which guarantee that they be bounded above on each compact subset of their domain [cf. F. Wolf, Bull. Amer. Math. Soc. **48** (1942), 925–932; J. London Math. Soc. **14** (1939), 208–216; MR **5**, 144; **1**, 48].

M. H. Heins (Urbana, Ill.)

11228:

Sakai, Akira. Some applications of the maximum principle for subharmonic functions. Proc. Japan Acad. **36** (1960), 332–334.

For a Riemann surface F of hyperbolic type and a fixed point p_0 on F, let $g(p, p_0)$ be the Green's function of F with pole at p_0. The modulus of a point p on F is defined by $\exp(-g(p, p_0))$. The author sketches the proofs of a few immediate analogues of theorems for analytic functions on the plane which concern the modulus of the functions. For functions defined on F and having values in another such surface $\hat{F}$, the author proves analogues of the Schwarz lemma, Hadamard three-circle theorem, Blaschke's interpolation theorem, etc.

G. Springer (Lawrence, Kans.)

11229:

Leis, Rolf. The influence of edges and corners on potential functions of surface layers. Arch. Rational Mech. Anal. **7** (1961), 212–223.

Let S be an analytic surface in 3-space which is bounded by a piecewise analytic curve, and let $U(p)$ denote the potential generated by the N-fold layer $\sigma(q)$ on S, i.e.,

$$U_N(p) = \int_S \sigma(q)\left(\frac{\partial}{\partial n_q}\right)^N \frac{dS_q}{r(p, q)},$$

where $\partial/\partial n$ is the normal derivative and $r(p, q)$ is the distance between p and q. By classical results, the value of $U_N(p)$ jumps in a specified manner if p crosses S in an interior point. The question investigated in the present paper is the behavior of $U_N(p)$ if p approaches a boundary point of S. It is shown that these boundary points are singularities of $U_N(p)$, and the behavior of the potential in the neighborhood of such points is described in detail.

Z. Nehari (Pittsburgh, Pa.)

11230:

Gilbert, R. P. Singularities of three-dimensional harmonic functions. Pacific J. Math. **10** (1960), 1243–1255.

A theorem due to Whittaker [Math. Ann. **57** (1903), 333–355] and Bergmann [Math. Z. **24** (1926), 641–669] allows one to represent each three-dimensional harmonic function $H(X)$ in the form

$$(1) \qquad H(X) = B_3(f, \mathscr{L}, X_0) = (2\pi i)^{-1} \int_{\mathscr{L}} f(t, u)/u\, du.$$

Here $f(t, u)$ is a function of two complex variables, $\mathscr{L}$ is a closed differentiable arc in the u-plane, $t = -(x-iy)u/2 + z + (x+iy)/2u$, $X = (x, y, z)$ and $X_0 = (x_0, y_0, z_0)$, and (1) holds in some neighborhood of X_0. For example, if $H(X)$ is regular in $|X| < \varepsilon$, then $H(X)$ has an expansion of the form

$$H(X) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{n,m} h_{n,m}(x, y, z),$$

where the $h_{n,m}$ are homogeneous polynomials of degree n in x, y, z chosen so that $t^n=\sum_{m=-n}^{n} h_{n,m}(x, y, z)u^{-m}$, and (1) holds with $\mathscr{L}$ the unit circle and

$$f(t, u) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{n,m}t^n u^m.$$

The author uses this representation to obtain relations between the singularities of $H(X)$ and $f(t, u)$. Theorem 1: If $S(x, y, z, u)=0$ is an implicit representation of the singularities of $u^{-1}f(t, u)$ and if $\mathscr{L}$ is the unit circle, then $H(X)$ is regular at $X=(x, y, z)$ if this point does not lie simultaneously on the two surfaces $S(x, y, z, u)=0$ and $\partial S(x, y, z, u)/\partial u=0$. The author derives an inverse for the Whittaker-Bergmann operator $B_3(f, \mathscr{L}, X_0)$ and Theorem 2 concerns a relation between the singularities of the harmonic function and of the function of two complex variables connected by this inverse. Similar results, connecting the singularities of axially symmetric harmonic functions with those of analytic functions, have been obtained by Szegö [J. Rational Mech. Anal. **3** (1954), 561–564; MR **16**, 34] and Nehari [same J. **5** (1956), 987–992; MR **18**, 293].

F. W. Gehring (Ann Arbor, Mich.)

11231:

Gilbert, Robert P. On the singularities of generalized axially symmetric potentials. Arch. Rational Mech. Anal. **6**, 171–176 (1960).

Generalized axially symmetric potentials are solutions of the partial differential equation $u_{zz}+u_{\rho\rho}+k\rho^{-1}u_\rho=0$. They can be represented in the form $u(r, \theta)=\int_{-1}^{1} f(\sigma)(u-1/u)^{k-1}/u\, du$, $\sigma=z+\frac{1}{2}i\rho(u+1/u)$, $z=r\cos\theta$, $\rho=r\sin\theta$. If $S(r, \theta|u)=0$ is the singularity manifold of the integrand in this integral representation, one of the two principal results of the paper states that u is regular at (r, θ) unless this point lies in the intersection of the two surfaces $S=0$ and $\partial S/\partial u=0$. The other principal result contains a corresponding statement about the singularities of $f(\sigma)$ in terms of the singular manifolds of u. The proof of these results is very similar to that of the corresponding results on three-dimensional harmonic functions [#11230] and is omitted.

A. Erdélyi (Pasadena, Calif.)

11232:

Green, John W. Functions that are harmonic or zero. Amer. J. Math. **82** (1960), 867–872.

The author's main theorem concerns real C' functions defined on a plane domain D. If, at each p in D, the function either vanishes or is harmonic, then the function must be harmonic on D. This result is an improved version (i.e., less restrictive hypotheses) of a theorem of Beckenbach [Proc. Amer. Math. Soc. **3** (1952), 765–769; MR **14**, 272].

An application is that no conductor potential in the strict sense can have continuous partial derivatives.

The author's methods also provide a new proof of the Radó-Behnke-Stein-Cartan theorem to the effect that a continuous mapping of a plane domain, D, into the plane which, at each point, either vanishes or is analytic, must be analytic on D.

W. C. Fox (New Orleans, La.)

11233:

Conlan, James; Diaz, J. B.; Parr, W. E. On the capacity of the icosahedron. J. Mathematical Phys. **2** (1961), 259–261.

The authors suggest a simple choice for the trial function to be used in the application of the Dirichlet principle to obtain upper bounds for the capacity of a regular solid. As a particular example they obtain bounds for the capacity of an icosahedron.

L. E. Payne (College Park, Md.)

11234:

Pommerenke, Christian. Einige Sätze über die Kapazität ebener Mengen. Math. Ann. **141** (1960), 143–152.

The principal result of the paper is the following. If $\mathfrak{E}$ is a closed, bounded (but not necessarily connected) point set in the complex plane with positive capacity (transfinite diameter), then $\mathfrak{E}$ can be enclosed by finitely many curves of total length $\Lambda<10.36$ cap $\mathfrak{E}$. (10.36 stands for any constant greater than $2\pi e^{1/2}$.) Also $\mathfrak{E}$ can be covered by finitely many circular discs the sum of whose radii is less than 2.59 cap $\mathfrak{E}$. ($2.59>(\pi/2)e^{1/2}$.) This result represents an extension of earlier ones of a similar nature, but concerning connected sets, which were obtained by the author in two previous papers [Math. Ann. **139** (1959), 64–75, 127–132; MR **22** #5740a, b]. The main tool used in the present paper, which seems of some interest in itself, is the following substitute for Riemann's mapping theorem, applied to the case of a multiply-connected region $\mathfrak{G}$. If it is assumed that ∞ is an interior point of $\mathfrak{G}$ and that the boundary of $\mathfrak{G}$ consists of finitely many closed Jordan curves, then it is possible, by removing finitely many curves from $\mathfrak{G}$, to obtain a simply-connected region $\mathfrak{G}_0$ and a function $g(z)$ which is univalent in $\mathfrak{G}_0$ with a simple pole at $z=\infty$; and the image of $\mathfrak{G}_0$ under the mapping $w=g(z)$ is a region obtained by removing from $|w|>1$ finitely many radial slits. This mapping function $g(z)$ is, in fact, obtained in such a way that $\log|g(z)|$ is the Green's function for the original region $\mathfrak{G}$.

It should be noted that, on page 148, the correct choice for δ should be $[(10.36/2\pi e^{1/2})-1]$ cap $\mathfrak{E}$.

F. Herzog (E. Lansing, Mich.)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

11235:

Urm, V. Ya. Some remarks on the asymptotic behavior of the solutions of difference equations. Dokl. Akad. Nauk SSSR **132** (1960), 56–59 (Russian); translated as Soviet Math. Dokl. **1**, 492–494.

Consider the difference equation

$$(*)\qquad u_k^{n+1} = \sum_{p=-\infty}^{\infty} c_{p-k}u_p^n,$$

where $\{c_k\}$ and $\{u_k^0\}$ are in l_2. If $\{u_k^n\}$ are the Fourier coefficients of $U^n(s)$, then (*) implies $U^{n+1}(s)=\lambda(s)U^n(s)$. Suppose that for $-\alpha\leqq s\leqq\alpha$, some $\alpha>0$,

$$\lambda(s) = \exp[i \sum_{j=1}^{2p-1} \beta_j s^j-\beta_{2p}s^{2p}+\beta_{2p+1}s^{2p+1}v(s)],$$

where the β_j are real, $|\lambda(s)|<1$ $(s\neq 0)$, $\beta_{2p}>0$, and $v(s)$ is analytic. If

$$\lambda'(s) = \exp[i \sum_{j=1}^{2p-1} \beta_j s^j-\beta_{2p}s^{2p}],$$

$-\alpha \leq s \leq \alpha$, $\lambda'(s)=0$, $|s|>\alpha$, then the author shows that the Fourier coefficients of $[\lambda'(s)]^n U^0(s)$ approximate asymptotically $\{u_k{}^n\}$ in the l_2 topology. Two examples are given. *B. F. Jones, Jr.* (Houston, Tex.)

11236:

Koehler, Fulton; Braden, Charles M. An oscillation theorem for solutions of a class of partial difference equations. Proc. Amer. Math. Soc. **10** (1959), 762–766.

Hadamard posed in 1908 the question whether a solution of the equation $\Delta^2 u(x,y)=p(x,y)$, with $u=\partial u/\partial n=0$ on the boundary and $p(x,y)\geq 0$, can oscillate in sign. It was answered in the affirmative first by Duffin and later by Garabedian, Szegö and the reviewer. The authors consider a similar question for difference equations of the general form

$$(1)\qquad Lu(x,y) \equiv \sum_{i=-r}^{r}\sum_{j=-m}^{m} c_{i,j}u(x+i,y+j) = p(x,y)$$

with constant coefficients $c_{i,j}$ satisfying the condition $c_{i,j}=c_{i,-j}$. The variables x and y are restricted to integers. Following the investigations by Duffin, the authors consider solutions of (1) in a strip S_N: $x\geq 0$, $0\leq y\leq N$ or in a full strip $S_N{}^*$: $0\leq y\leq N$, setting $u(x,y)=0$ for $-m\leq y\leq 0$ and $N<y\leq N+m$, and leaving $u(x,y)$ arbitrary in the first case for $-1\leq x<0$, $0\leq y\leq N$. By calling $u(x,i)=u_i(x)$, equation (1) can be replaced by a system of equations

$$(2)\qquad \sum_{j=0}^{N} Q_{i-j}u_j = 0 \qquad (i=0,\cdots,N)$$

with $Q_k(|k|\leq N)$ representing operators in x alone. Let $\Delta(z)=\det Q_{i-j}(z)|_{i,j=0}^N$. The fundamental results of the paper can be expressed in the following two theorems. (A) If the roots of $\Delta(z)=0$ are all non-real, then a non-trivial solution of (1) in S_N with $p=0$ is such that each $u_i(x)$ changes its sign infinitely often as $x\to\infty$. (B) If the roots of $\Delta(z)=0$ are all non-real and $p\geq 0$ in $S_N{}^*$, but $=0$ for $x\geq 0$, then any solution of (1) in $S_N{}^*$ is such that each u_i changes its sign infinitely often as $x\to\infty$.
C. Loewner (Stanford, Calif.)

11237:

Berezanskiĭ, Yu. M. Energy inequalities for some classes of equations of mixed type. Dokl. Akad. Nauk SSSR **132** (1960), 9–12 (Russian); translated as Soviet Math. Dokl. **1**, 447–451.

The energy inequalities cited in the author's earlier paper in same Dokl. **131** (1960), 478–481 [MR **22** #9720] are proved when the operator $\mathscr{L}[u]$, here given by

$$\mathscr{L}[u] = \sum_{j,k=1}^{3} D_j(a_{jk}(x)D_ku) + \sum_{j=1}^{2} a_j(x)D_ju + a(x)u \qquad (D_j = \partial/\partial x_j,\ j=1,2),$$

is of mixed type in a bounded domain of the (x_1, x_2)-plane. The essential step in the proof is once again the obtaining of an identity by partial integration of an integral of Green's type. *R. N. Goss* (San Diego, Calif.)

11238:

Radó, François. Équations fonctionnelles caracterisant les nomogrammes avec trois échelles rectilignes. Mathematica (Cluj) **1 (24)** (1959), no. 1, 143–166.

Translation of a Romanian original [Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **9** (1958), 249–319; MR **21** #6710].

SEQUENCES, SERIES, SUMMABILITY

See also 10947, 11252.

11239:

Parameswaran, M. R. On the translativity of Hausdorff- and some related methods of summability. J. Indian Math. Soc. (N.S.) **23** (1959), 45–64 (1960).

The summation processes (H,μ_n), (H^*,μ_n) and (S^*,μ_n) are defined respectively by the transformations

$$\tau_n = \sum_{k=0}^{n}\binom{n}{k}\Delta^{n-k}\mu_k\sigma_k,$$

$$\tau_n{}^* = \sum_{k=n}^{\infty}\binom{k}{n}\Delta^{k-n}\mu_n\sigma_k,$$

$$\lambda_n = \sum_{k=0}^{\infty}\binom{n+k}{k}\Delta^k\mu_n\sigma_k.$$

The paper deals mostly with the translativity properties of the above three summation processes; various concepts such as left translativity, right translativity and absolute translativity are defined and several results, some of which are refinements or generalizations of the results of Kuttner [Quart. J. Math. Oxford Ser. (2) **8** (1957), 272–278; MR **20** #7168] and of the reviewer [ibid., 197–213; Proc. Nat. Inst. Sci. India Part A **24** (1958), 4–14; MR **20** #7167, 4117] are proved. The author characterizes the above properties in terms of the generating function $\chi(t)$ associated with the (moment) sequence μ_n in the above processes. A lemma of importance [and of significance in itself] is that for suitable classes of sequences, $\lim_{n\to\infty}|t_n-t_{n-1}|=0$, where t_n is any one of the sequences τ_n, $\tau_n{}^*$ and λ_n. Using this result the author proves that if (H,μ_n) is a conservative Hausdorff method with $\lim \mu_n\neq 0$ and if $\{\sigma_n\}$ is such that $\{\sigma_n-\sigma_{n-1}\}$ is Borel summable to l and $\{\sigma_n-\sigma_{n-1}\}=O(1)$, then $\{\sigma_n-\sigma_{n-1}\}\to l$. The translativity properties of the other two methods are also investigated in detail. Finally, as an application, he proves that both the methods (H,μ_n) and (H^*,μ_n) have the property that if they are conservative then they sum almost all sequences of 0's and 1's or no divergent sequence of 0's and 1's according as $\lim \mu_n=0$ or $\neq 0$. In a short review of this kind it is difficult to summarize completely the achievements of the author in this direction, but the above results typify them. *M. S. Ramanujan* (Ann Arbor, Mich.)

11240:

Chen, Yung-ming. On products of power series. Monatsh. Math. **64** (1960), 317–320.

The author obtains a stronger form of an Abelian-type theorem dealing with Nörlund summability [A. M. Ostrowski, Compositio Math. **14** (1959), 41–49; MR **21** #5840] by weakening the hypotheses in an essential way. Let $f(z)=\sum s_nz^n$, $F(z)=\sum S_nz^n$, $\psi(z)=\sum T_nz^n$, where $S_n\geq 0$, $T_n\geq 0$ and $\sum S_n=\infty$. Let $f(z)\psi(z)=\sum r_nz^n$ and $F(z)\psi(z)=\sum R_nz^n$, and assume that $\liminf T_n/T_{n-k}>r>0$ for any positive or negative choice of k. Then, $\liminf s_n/S_n\leq \liminf r_n/R_n\leq \limsup r_n/R_n\leq \limsup s_n/S_n$.
R. C. Buck (Princeton, N.J.)

11241:

Turowicz, A. B. Sur l'approximation des racines de nombres positifs. Ann. Polon. Math. 8 (1960), 265–269.

Let $A>0$ and $x_0>0$ be arbitrary. It is shown that the sequence defined by

$$x_{k+1}=\frac{(n-1)x_k^{n+1}+(n+1)Ax_k}{(n+1)x_k^n+(n-1)A},$$

$n\geqq 2$, n a positive integer, converges to $A^{1/n}$.

E. Frank (Chicago, Ill.)

11242:

Mikolajska, Z. Remarque sur la note de A. B. Turowicz sur l'approximation des racines de nombres positifs. Ann. Polon. Math. 8 (1960), 285–289.

The method used by A. B. Turowicz [#11241] to show the convergence of $A^{1/n}$ is generalized to the sequence $\{x_k\}$ defined by

$$x_k=\frac{x_{k-1}\{\psi(x_{k-1})-\phi(x_{k-1}^n-A)\}}{\psi(x_{k-1})+\phi(x_{k-1}^n-A)},\ x_0>0,$$

where the functions $\psi(x)$ and $\phi(u)$ are continuous and satisfy the relations

$$\psi(x)+\phi(x^n-A)>0 \quad \text{for } x\geqq 0,$$

$$\psi(x)-\frac{x+A^{1/n}}{x-A^{1/n}}\phi(x^n-A)>0 \quad \text{for } x^n\neq A,$$

$$\phi(u)=-\phi(-u),\ \phi(u) \text{ montone.}$$

E. Frank (Chicago, Ill.)

11243:

Borwein, D. An extension of a theorem on the equivalence between absolute Rieszian and absolute Cesàro summability. Proc. Glasgow Math. Assoc. 4, 81–83 (1959).

Für $p\geqq 1$, $k>1-1/p$, q reell heisse $\sum a_n|R,k,q|_p$-summierbar, wenn gilt

$$\int_1^\infty u^{pq+p-1}\frac{d}{du}|C_k(u)|^p\,du<\infty,$$

$$C_k(u)=u^{-k}\sum_{n<u}(u-n)^k a_n.$$

Für $k\geqq q-1/p$ ist $|R,k,q|_p$-Summierbarkeit mit der von T. Flett [Proc. London Math. Soc. (3) 8 (1958), 357–387; MR 21 #1481] eingeführten $|C,k,q|_p$-Summierbarkeit äquivalent.

A. Peyerimhoff (Marburg)

11244:

Keogh, F. R.; Petersen, G. M. Riesz summability of subsequences. Quart. J. Math. Oxford Ser. (2) 12 (1961), 33–44.

For a sequence s_n and a real number t, $0\leqq t\leqq 1$, $s_n(t)$ denotes the subsequence obtained by omitting from the original sequence those terms s_k for which the kth digit of the dyadic expansion of t is zero. Let (R,p_n) be a regular Riesz means which satisfies $p_{n+1}\geqq p_n>0$, $P_n=p_0+\cdots+p_n$. The main results are the following. (a) If $\sum(p_n/P_n)^2<+\infty$ and $p_nP_k/p_kP_n\leqq\text{const}$ for $n=0,1,\cdots,0\leqq k\leqq 3n$, and if s_n is (R,p_n)-summable and bounded (or satisfies only $\sum(s_np_n/P_n)^2<+\infty$), then almost all subsequences $s_n(t)$ are (R,p_n)-summable to the same sum. (b) If p_n/P_n decreases to zero, if $P_np_k/P_kp_n\leqq\text{const}$ for $\frac{1}{3}n\leqq k\leqq 3n$, and if s_n if not bounded but (R,p_n)-summable, then almost all subsequences $s_n(t)$ are themselves not (R,p_n)-summable. Buck and Pollard [Bull. Amer. Math. Soc. 49 (1943), 924–931; MR 5, 117] proved similar results for the $(C,1)$-method. Further remarks compare the property of a method asserted in (a) with the Borel property.

G. G. Lorentz (Syracuse, N.Y.)

11245:

Parter, Seymour V. Extreme eigenvalues of Toeplitz forms and applications to elliptic difference equations. Trans. Amer. Math. Soc. 99 (1961), 153–192.

Let f be real, belong to $L(-\pi,\pi)$, and have Fourier coefficients C_j. The $n\times n$ matrix with i,j entry C_{i-j} is a finite Toeplitz matrix. Let its eigenvalues be $\lambda_{1,n}\leqq\cdots\leqq\lambda_{n,n}$. The problem is the determination of the asymptotic behavior of $\lambda_{\nu,n}$ (for fixed ν) as $n\to\infty$. It was shown by Kac, Murdock, and Szegö [J. Rational Mech. Anal. 2 (1953), 767–800; MR 15, 538] that if f is sufficiently smooth, assumes its minimum m at one point only (say at $\theta=0$) and $f(\theta)-m$ has a zero of order two at $\theta=0$ then $\lambda_{\nu,n}=f(\nu\pi n^{-1}+o(n^{-1}))$. The author generalizes this in several directions: the order of the zero of $f(\theta)-m$ may be four rather than two; f may attain its minimum at two points; f may be a function taking Hermitian matrix values so that each C_j is a matrix. The results obtained are used to investigate the rates of convergence of certain iterative schemes for elliptic difference equations on rectangular domains.

H. Widom (Ithaca, N.Y.)

11246:

Gyires, Béla. Über Toeplitzsche Determinanten, welche zu einer reelle Funktionenmatrix gehören. Acta Univ. Debrecen 5 (1958), 145–158 (1959). (Hungarian. German summary)

Let $f(x)=[f_{ij}(x)]$ be a $p\times p$ matrix of 2π-periodic real-valued functions of the real variable x. f is said to be (A) if for some $d>0$ and all real $x_1,\cdots,x_p$, all principal minors of $[f_{ij}(x_j)]$ are not less than d. The author proves that for a bounded $f(x)$, $I+\alpha f(x)$ and $(I+\alpha f(x))^{-1}$ are (A) provided α is sufficiently small and positive (I is the $p\times p$ unit matrix). If f is continuous and (A) [if f^{-1} is also (A)], then f can be approximated uniformly by a trigonometric matrix polynomial which is (A) [and whose inverse is also (A)].

From now on let $f(x)$ be integrable, form the $p\times p$ matrices $c_k=(2\pi)^{-1}\int_{-\infty}^{\infty}f(x)e^{-ikx}dx$ $(k=0,\pm1,\pm2,\cdots)$ and the $(n+1)p\times(n+1)p$ Toeplitz hypermatrices $T_n(f)=[c_{j-i}]$, $i,j=1,2,\cdots,n+1$. Also set $D_n(f)=\det T_n(f)$. If f is symmetric, integrable, and (A), then $T_n(f)$ is positive definite Hermitian, and $D_n(f)/D_{n-1}(f)$ approaches a limit, $D(f)$, as $n\to\infty$. If, moreover, $0<m\leqq\det[f_{ij}(x_j)]\leqq M$ for all real $x_1,\cdots,x_p$, then $m^{n+1}\leqq D_n(f)\leqq M^{n+1}$. If f is continuous, symmetric and (A), then $D(f)\geqq G(f)=\exp(2\pi)^{-1}\int_{-\pi}^{\pi}\log\det f(x)dx$; and if f is continuous, symmetric and positive definite, then $D(f)=G(f)$.

Let $\lambda_k(x)$, $k=1,\cdots,p$, be the latent roots of $f(x)$ and $\lambda_{\nu k}^{(n)}$, $\nu=0,1,\cdots,n$; $k=1,\cdots,p$, those of $T_n(f)$. If f is continuous and symmetric, the $\lambda_k(x)$ are chosen to be continuous, and if $m\leqq\lambda_k(x)\leqq M$, $k=1,\cdots,p$; then also $m\leqq\lambda_{\nu k}^{(n)}\leqq M$ for $\nu=0,1,\cdots,n$; $k=1,\cdots,p$; $n=1,2,\cdots$ and for every continuous function F on $m\leqq\lambda\leqq M$,

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} \sum_{k=1}^{p} F(\lambda_{\nu k}{}^{(n)}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{k=1}^{p} F(\lambda_k(x))\, dx.$$

Both results and proof are generalizations of Szegö's work for $p=1$ [G. Szegö, Math. Ann. **76** (1915), 490–503; U. Grenander and G. Szegö, *Toeplitz forms and their applications*, Univ. of California Press, Berkeley, Calif., 1958; MR **20** #1349; § 5.2]. *A. Erdélyi* (Pasadena, Calif.)

11247:

König, Heinz. Neuer Beweis eines klassischen Tauber-Satzes. Arch. Math. **11** (1960), 278–279.

Es handelt sich um einen neuen Beweis des klassischen Tauber-Satzes: Ist $\varphi(t)$ für $t \geq 0$ monoton wachsend und existiert $f(x) = \int_0^\infty e^{-xt} d\varphi(t)$ für $x > 0$, aus $f(x) \sim A/x^\lambda$, $x \to 0+$, $\lambda > 0$ folgt dann $\varphi(t) \sim At^\lambda/\Gamma(\lambda+1)$, $t \to \infty$.

Der Gedankengang ist der folgende: Aus der Monotonie von $\varphi(t)$ schliesst man elementar (für $\alpha \geq 1$)

$$0 \leq \varphi(\alpha t)/\alpha^\lambda \leq M \quad \text{für } 0 \leq t \leq 1,$$
$$\leq Mt^\lambda \text{ für } 1 \leq t.$$

Nach dem Auswahlsatz von Helly gibt es dann eine Folge $\alpha_j \nearrow \infty$ für die $C = \lim_{j\to\infty} \varphi(\alpha_j)/\alpha_j^\lambda$ existiert. Durch sukzessive Anwendung des Hellyschen Satzes beweist man jetzt $C = A/\Gamma(\lambda+1)$. Monotonie von $\varphi(t)$ ergibt dann die Behauptung.

Der Verfasser betont dass ähnliche Ansätze schon in der berühmten Thesis von R. Schmidt [Math. Z. **22** (1925), 89–152], und bei A. Beurling [Acta Math. **77** (1945), 127–136; MR **7**, 61] zu finden sind.

V. Vučković (Belgrade)

APPROXIMATIONS AND EXPANSIONS

11248:

Steinberg, Jacob. Classes of biorthonormal systems. Ann. Mat. Pura Appl. (4) **52** (1960), 183–218.

In earlier work [Bull. Res. Council Israel Sect. F **7F** (1957/58), 55–68; MR **21** #2167] the author examined the integral equation (1) $f(s) - \lambda \int_{-\infty}^{\infty} K(bs-t) f(t) dt = 0$, where b is a parameter and K satisfies: (i) $|K(x)| < Ce^{-|h|x}$, $h > 0$, $-\infty < x < \infty$; (ii) $\int_{-\infty}^{\infty} K(x) dx = \lambda_0^{-1} \neq 0$. He found the eigenvalues $\lambda = \lambda_0 b^{-n}$ $(n = 0, 1, \cdots)$, and corresponding eigenfunctions $\{p_n(t)\}$ that form an Appell set of polynomials. Now it is shown that if $b > 1$ and if K' satisfies a condition of type (i) above, then (1) also has the eigenvalues $\lambda = \lambda_0 b^n$ $(n = 1, 2, \cdots)$, with eigenfunctions $q^{(n-1)}(t)$, where $q(t) = (1/2\pi i) \int_{-i\infty}^{i\infty} e^{ut} A(u) du$ and $A(t)$ is the generating function for $\{p_n(t)\}$. Let $\{p_n^*(t)\}$ be the Appell eigenfunction set for the transpose of equation (1), and set $q_n(t) = (-1)^n q^{(n)}(t)$. Then the biorthogonal relation $\int_{-\infty}^{\infty} p_m^*(t) q_n(t) dt = \delta_{mn}$ holds.

The above results suggest two problems: (a) Given an Appell set $\{p_n(t)\}$, to find conditions on its generating function $A(t)$ in order that an infinitely differentiable function $q(t)$ exist such that $\{q_n(t)\}$ and $\{p_n(t)\}$ be biorthogonal. (b) Conversely, for what $q(t)$ will there correspond an Appell set $\{p_n(t)\}$? Sufficient conditions are obtained for which the respective answers are affirmative. Under certain conditions, the Fourier series $f(t) \sim \sum_0^\infty d_n q_n(t)$ is shown to be B^2-summable; and completeness of $\{q_n(t)\}$ in $L_2(-\infty, \infty)$ is examined.

I. M. Sheffer (University Park, Pa.)

11249:

Schoenberg, I. J. On the question of unicity in the theory of best approximation. Ann. New York Acad. Sci. **86**, 682–692 (1960).

Let S be a compact Hausdorff space and $C[S]$ the algebra of continuous real-valued functions on S. If $f_1, f_2, \cdots, f_n$ and F are in $C[S]$, then there are coefficients $a_1, \cdots, a_n$ such that $\|f - \sum a_i f_i\|$ is a minimum, where $\|g\| = \max_{x\in S} |g(x)|$. In this expository paper the author gives a detailed proof (in this more general setting) that the best approximating "polynomial" $\sum a_i f_i$ is unique if and only if $D(x_1, x_2, \cdots, x_n) = \det[f_j(x_i)] \neq 0$ for all choices of $x_1, \cdots, x_n$ as distinct points of S. It is known [Mairhuber, Proc. Amer. Math. Soc. **7** (1956), 609–615; MR **18**, 125; Buck, *On numerical approximation* (Proceedings of a symposium, Madison, 1958), pp 11–23, Univ. of Wisconsin Press, Madison, Wis., 1959; MR **21** #420] that this condition in turn forces S to be a subset of an interval or a circumference. In an afterword by Rivlin, the existence of examples is revealed which show that in the case of several variables, even smoothness of F does not prevent the existence of many "best" approximations.

R. C. Buck (Madison, Wis.)

11250:

Mergelyan, S. N. Best approximations with a weight on the straight line. Dokl. Akad. Nauk SSSR **132** (1960), 287–290 (Russian); translated as Soviet Math. Dokl. **1**, 552–556.

Let

$$h(x) = h(0) \exp\left\{-\int_0^{|x|} \frac{\omega(t)}{t}\, dt\right\},$$

$$\int_{-\infty}^{+\infty} \frac{\ln h(\xi)}{1+\xi^2}\, d\xi = -\infty,$$

where $\omega(t) \geq 0$ increases monotonically. Let us denote by $E_n(h, f)$ the lower bound of

$$\sup_{-\infty < x < +\infty} h(x)|f(x) - P_n(x)|$$

where $P_n(x)$ runs through the polynomials of degrees not exceeding n. The author obtains an upper bound for $E_n(h, (x-a)^{-1})$, Im $a \neq 0$, and states that it can be applied to estimate the best weighted approximation of functions characterized by some differential property.

G. Freud (Budapest)

11251:

Fullerton, R. E. The structure of contours of a Fréchet surface. Illinois J. Math. **4** (1960), 619–628.

The smoothing method for contours developed by Cesari and the author [same J. **1** (1957), 395–405; MR **19**, 844] is used to prove the following theorem. Let S be a nondegenerate surface of the type of the 2-cell, T a light representation of S on the unit square Q, and f a real-valued continuous function on $[S]$. Let $\{\gamma\}_f$ be the set of all components of contours in Q whose images on S have finite length. Then, except for a countable subset of $\{\gamma\}_f$, every element of $\{\gamma\}_f$ is a point, a simple arc, or a simple closed curve. This improves considerably an earlier result

of the author [Riv. Mat. Univ. Parma **4** (1953), 207–212; MR **15**, 612]. *W. H. Fleming* (Providence, R.I.)

FOURIER ANALYSIS

See also 11149, 11245, B11425.

11252:

Petersen, G. M. Summability of a class of Fourier series. Proc. Amer. Math. Soc. **11** (1960), 994–998.

Let $A=(a_{n,k})$ be a triangular matrix such that

$$\lim_{n\to\infty}\sum_k k^r|a_{n,k}-a_{n,k+1}| = 0$$

for some r, $0<r<1$, and let $f(t)$ be an integrable function such that $f(x+h)-f(x)=o(1/|\log|h||)$ for some x as $h\to 0$. Then the Fourier series of f is (A)-summable to $f(x)$ at the point x, that is,

$$\frac{2}{\pi}\sum_{k=1}^{n} a_{n,k}\int_0^\pi f(x+t)\,\frac{\sin(k+1/2)t}{2\sin t/2}\,dt \to f(x).$$

The theorem does not hold for $r=0$. *S. Izumi* (Sendai)

11253:

Hsiang, Fu Cheng. On some extensions of theorems of Fejér. Math. Scand. **7** (1959), 333–336.

Let f be a Lebesgue-integrable function of bounded variation on $[0, 2\pi]$ and of period 2π. Let $\Omega=\{\omega_{ij}\}$ be a Toeplitz matrix with $\omega_{ij}=0$, $i<j$. If

$$\sum_{j=0}^{n}|\omega_{n,j}-2\omega_{n,j+1}+\omega_{n,j+2}| = o(1) \text{ as } n\to\infty,$$

then the derived series of the Fourier series of f is summable (Ω) to $\pi^{-1}\{f(x+0)-f(x-0)\}$. The above theorem contains a theorem of Siddiqi [Math. Z. **61** (1954), 79–81; MR **17**, 475], in which the same conclusion is derived from the hypothesis $\sum_{j=0}^{n}|\omega_{n,j}-\omega_{n,j+1}|=o(1)$. A special case is a result due to Fejér [J. Reine Angew. Math. **142** (1913), 165–188] concerning summability (C, r), $r>0$. *P. Civin* (Gainesville, Fla.)

11254:

Goldberg, Richard R. ★Fourier transforms. Cambridge Tracts in Mathematics and Mathematical Physics, No. 52. Cambridge University Press, New York, 1961. viii+76 pp. $3.75.

The following topics are treated in this short book: The L^1-Fourier transform (including Wiener's theorem), the Plancherel theorem, and Bochner's theorem on positive-definite functions. The action takes place on the real line, but an appendix contains a brief outline of generalizations. No knowledge of functional analysis is required of the reader. *W. Rudin* (Madison, Wis.)

11255:

Loomis, L. H. The spectral characterization of a class of almost periodic functions. Ann. of Math. (2) **72** (1960), 362–368.

On note G un groupe abélien localement compact, $\hat{G}$ son dual, $\varphi\in L^\infty(G)$, $\mathrm{sp}(\varphi)=$ spectre de $\varphi=$ ensemble des caractères approchables dans $L^\infty(G)$ faible par les combinaisons linéaires de translatées φ_s de φ. Les résultats principaux sont les suivants: si $\mathrm{sp}(\varphi)$ est compact et résiduel, φ est presque-périodique; si $\mathrm{sp}(\varphi)$ est résiduel et si φ est uniformément continue, φ est presque-périodique. Les cas particuliers sont connus [B. Lewitan, Bull. Amer. Math. Soc. **43** (1937), 677–679; H. J. Reiter, Trans. Amer. Math. Soc. **75** (1953), 505–509; MR **15**, 881]. Inversement, tout parfait non vide sur $\hat{G}$ contient le spectre d'une φ non. p.p. L'auteur introduit la notion de presque-périodicité en un point $\alpha_0\in\hat{G}$: φ est dit p.p. en α_0 s'il existe $f\in L^1(G)$ tel que $f*\varphi$ est p.p. et $\hat{f}(\alpha_0)\neq 0$. Il montre que si $\mathrm{sp}(\varphi)$ est compact et si φ est p.p. en chaque point de $\hat{G}$, φ est p.p.; si $\varphi-\varphi_s$ est p.p. pour toute translation $s\in G$, φ est p.p. *J.-P. Kahane* (Orsay)

11256:

Koosis, Paul J. Les sous-espaces intérieurement compacts de $L_2(0,\infty)$. Sèminaire P. Lelong, 1958/59, exp. 9, 4 pp. Faculté des Sciences de Paris, 1959.

Pour $f\in L_2(0,+\infty)$, et $h\geqq 0$, on pose: $f^h(x)=f(x+h)$, pour $x\geqq 0$ et $f^h(x)=0$, pour $x<0$; on appelle T_h l'application de $L_2(0,+\infty)$ dans lui-même définie par: $T_hf=f^h$. Un sous-espace linéaire $E\subset L_2$ est dit intérieurement compact si $T_hE\subset E$, pour $h\geqq 0$, et si T_h est complètement continua, quel que soit h, lorsqu'on le considère comme une application linéaire de E dans lui-même.

On pose $\Lambda=\{\lambda_n, p_n+1\}$ $(n=1, 2, \cdots)$, où les λ_n sont des nombres complexes tels que $\Im\lambda_n>0$, les p_n des entiers $\geqq 0$; on appelle $L_2(\Lambda)$ le sous-espace linéaire de $L_2(0,+\infty)$ engendré par les fonctions: $x^re^{i\lambda_n x}$ $(r=0, 1, \cdots, p_n)$. On a le théorème suivant: pour que E soit intérieurement compact, il faut et il suffit que ce soit un $L_2(\Lambda)$ tel que: $\Im\lambda_n\to+\infty$, et que: $\sum_1^\infty (p_n+1)\Im\lambda_n/|\lambda_n-t|^2\to 0$ pour $t\to\pm\infty$. *J. Favard* (Paris)

11257:

Kahane, Jean-Pierre. Fonctions qui opèrent dans les algèbres de transformées de Fourier de suites, de fonctions ou de mesures sommables. Séminaire P. Lelong, 1958/59, exp. 5, 6 pp. Faculté des Sciences de Paris, 1959.

This is a brief exposition of the results obtained by Katznelson [C. R. Acad. Sci. Paris **247** (1958), 404–406; MR **20** #4152], Helson and Kahane [ibid. **247** (1958), 626–628; MR **21** #4737], and Kahane and Rudin [ibid. **247** (1958), 773–775; MR **21** #1488]. Extensions to other groups have in the meantime been published by Helson, Kahane, Katznelson and Rudin in Acta Math. **102** (1959), 135–137 [MR **22** #6980]. *W. Rudin* (Madison, Wis.)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See also 11270, 11271, B11432.

11258:

Iwata, Giiti. Applications of Mellin transforms to some problems of statistical mechanics. Progr. Theoret. Phys. **24** (1960), 1118–1122.

The Mellin transform is used to obtain the asymptotic form of certain integrals containing power series where term-by-term integration is not permissible. After expressing the power series in terms of suitable Mellin integrals, e.g.,

$$\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\pi}{z \sin \pi z} x^z\,dz$$

with $1<\sigma<2$, and closing contours suitably in the complex plane, the desired asymptotic form is exhibited in terms of residues. In this way some results in the statistical mechanics of the electron gas and hard spheres are obtained in simpler ways than hitherto.

D. Falkoff (Waltham, Mass.)

11259:

Guy, Douglas L. Hankel multiplier transformations and weighted p-norms. Trans. Amer. Math. Soc. **95** (1960), 137–189.

Given an integral transform I on a function space and a linear transformation T in the function space, T is called an I-multiplier transform provided that there exists a function $\phi(y)$ such that $Tf(x)=I^{-1}(\phi(y)If(x))$. The present paper is concerned with Hankel multiplier transformations in L^p spaces, where $1<p\leq 2$. The author is able to establish for these transformations a close analogue of a theorem of J. Marcinkiewicz [Studia Math. 8 (1939), 78–91] on a criterion for the existence of bounds for L^p Fourier series multiplier transformations. As a special case of this result, one has G. M. Wing's result [Pacific J. Math. **1** (1951), 313–319; MR **13**, 342] on L^p convergence of the partial integrals defining the inversion formula for Hankel transforms.

D. S. Greenstein (Evanston, Ill.)

11260:

Kalisch, G. K. On isometric equivalence of certain Volterra operators. Proc. Amer. Math. Soc. **12** (1961), 93–98.

The conditions derived in the last section of a previous paper [Ann. of Math. (2) **66** (1957), 481–494; MR **19**, 970] for isometric equivalence of two Volterra transformations of the form $T_F=\int_x^1 F(x,y)f(y)dy$ with $F(x,y)=a(y-x)^{m-1}G(x,y)$, $|a|=1$, m a positive integer, on $L_2[0,1]$ to $L_2[0,1]$ are extended to transformations on $L_p[0,1]$ to $L_p[0,1]$ for $1<p<\infty$, where it is assumed that the only reducing manifolds of the operators are the spaces $L_p[0,c]$, c in $[0,1]$ for the same p.

T. H. Hildebrandt (Ann Arbor, Mich.)

11261:

Kalisch, G. K. On similarity invariants of certain operators in L_p. Pacific J. Math. **11** (1961), 247–252.

Let $T_F(f(x))=\int_x^1 F(x,y)f(y)dy$ (with $F(x,y)=(y-x)^{m-1}aG(x,y)$, m a positive integer, a complex with $|a|=1$) be a transformation on $L_p[0,1]$ to $L_p[0,1]$, $p>1$. The purpose of this note is to extend Theorem 2 and its corollary of a previous paper [Ann. of Math. (2) **66** (1957), 481–494; MR **19**, 970], proved for $a=1$, to the case where a is any complex number with $|a|=1$; specifically, if either (1) $G(x,y)$ is analytic in a suitable region, or (2) $G(x,y)=G(y-x)$ with $G(0)\neq 0$ and in C^2, or (3) G is in C^2, $G(x,x)>0$ and $m=1$, then for any complex number A, the operator $AI+T_F$ is similar to a unique operator of the form $AI+caT_E{}^m$, where

$$T_E{}^m f = (1/(\Gamma(m)))\int_x^1 (y-x)^{m-1}f(y)\,dy.$$

T. H. Hildebrandt (Ann Arbor, Mich.)

11262:

Buschman, R. G. A note on a convolution. Amer. Math. Monthly **67** (1960), 364–365.

An integral transform related to the Laplace transform, but based on $f(s)=s\int_1^\infty x^{-s-1}F(x)dx$, is introduced. Its convolution relation is given, along with another relation involving Dirichlet series. These relations are used together to produce a formula, whose special forms are identities involving sums, and integrals of sums, of logarithms, the Möbius function, and Euler's function.

T. E. Hull (Vancouver, B.C.)

11263:

Zaidman, S. Sur la représentation des fonctions vectorielles par des intégrales de Laplace. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2 (50)** (1958), 479–483.

This paper is concerned with the following representation theorem proved by I. Miyadera [Tôhoku Math. J. (2) **8** (1956), 170–180; MR **18**, 748], which generalizes a theorem due to Widder: Let $f(s)$ be a vector-valued function defined for $s>0$, with values in a reflexive Banach space X. Necessary and sufficient conditions for the existence of a vector-valued function $\varphi(t)$ defined for $t\geq 0$ with values in X, belonging to $B_\infty[(0,\infty);X]$ and such that $f(s)=\int_0^\infty \varphi(t)\exp(-st)dt$, are (i) $f(s)$ is indefinitely strongly differentiable for $s>0$, and (ii) the set of elements $s^{k+1}f^{(k)}(s)/k!$, $k=0,1,2,\cdots$, $s>0$, is bounded. The question arises as to whether reflexivity is necessary in the hypothesis of this theorem. The author shows that if X is any Banach space in which (i) and (ii) suffice for the indicated representation of $f(s)$, then for each bounded linear transformation T on $L_1(0,\infty)$ to X there exists a vector-valued function $\varphi(t)$ in $B_\infty[(0,\infty);X]$ such that $Tg=\int_0^\infty g(t)\varphi(t)dt$ for all $g\in L_1(0,\infty)$. Using this result, it is shown that for $X=L_1(0,\infty)$ the conditions (i) and (ii) are not sufficient in general for the representation.

R. S. Phillips (Stanford, Calif.)

11264:

Malgrange, Bernard. Sur les équations de convolution. Séminaire P. Lelong, 1958/59, exp. 15, 8 pp. Faculté des Sciences de Paris, 1959.

Soit $\mathscr{D}'$ resp. $\mathscr{E}'$ l'espace des distributions resp. distributions à support compact sur R^n. L'auteur donne une démonstration simplifiée du théorème {qu'il a établi dans sa thèse [Ann. Inst. Fourier Grenoble **6** (1955/56), 271–355; MR **19**, 280]}: quel que soit $\mu\in\mathscr{E}'$, la sous-espace $(f\colon \mu * f=0)$ de $\mathscr{D}'$ est engendré topologiquement par les exponentielles-polynômes qu'il contient. La démonstration est basée sur le lemme [proposée par J.-P. Kahane dans cours professé au Tata Institute, Bombay, 1957]: Soient ν, $\mu\in\mathscr{E}'$ telles que le quotient des transformées de Fourier $\hat{\nu}/\hat{\mu}$ soit une fonction entière et soit P un polynôme; l'équation $(P\mu)*\nu=\mu*\nu_P$ admet toujours une solution $\nu_P\in\mathscr{E}'$. L'auteur donne deux compléments du lemme: (1) L'enveloppe convexe du support de ν_P est contenue dans celle de ν; ceci entraîne le théorème de supports [cf. J.-L. Lions, J. Analyse Math. **2** (1953), 369–380; MR **15**, 307]. (2) Si l'on a de plus $\nu\in\mathscr{K}^s$ (où $\mathscr{K}^s$ désigne l'espace des distributions à support compact dont les dérivées d'ordre $\leq s$ sont dans L^2), alors $\nu_P\in\mathscr{K}^{s-k}$, où k ne dépend que de μ et P; l'auteur indique une conséquence {qui généralise un théorème d'Ehrenpreis [Amer. J. Math. **77** (1955), 286–292; MR **16**, 1123]}: quel que soit $\mu\in\mathscr{E}'$,

$g \in \mathscr{L}^s$ (où $g \in \mathscr{L}^s \Leftrightarrow \alpha g \in \mathscr{K}^s$ pour tout α indéfiniment dérivable à support compact) et un polynôme P, l'équation $\mu * f = (P\mu) * g$ admet une solution $f \in \mathscr{L}^{s-k'}$, où k' ne dépend que de μ et P. *S. Łojasiewicz* (Kraków)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See 11260, 11261.

FUNCTIONAL ANALYSIS

See also 11067, 11074, 11089, 11194, 11218, 11264, 11337, B11427.

11265:

Schaefer, Helmut. Halbgeordnete lokalkonvexe Vektorräume. III. Math. Ann. **141** (1960), 113–142.

The present paper is the third installment of the author's exhaustive study of partially ordered linear spaces. The terms which are introduced in the reviews of the earlier papers [Math. Ann. **135** (1958), 115–141; **138** (1959), 259–286; MR **21** #5134, 5135] will be used freely.

Section 12 presents a study of spectra of non-bounded positive operators. A closed non-bounded linear operator T in a partially ordered Banach space is called positive if there exists a sub-cone K_0 of the positive cone K such that the closure of $T[K_0]$ is equal to K. As in the bounded case $\rho(T)$ is the set of all complex numbers λ such that $\lambda - T$ has a bounded inverse, and $\sigma(T)$ denotes the complement of $\rho(T)$ in the complex plane. If E is a partially ordered Banach space with closed, normal, and strict BZ-(positive) cone, and if T is a (not necessarily bounded) positive operator in E that sends a sub-cone K_0 of K, which generates the domain of T, into K in an one-to-one manner, then $0 \in \rho(T)$ and either $\sigma(T)$ is empty or $\inf\{|\lambda|: \lambda \in \sigma(T)\} \in \sigma(T)$. If T is an unbounded densely defined operator in a partially ordered Banach space with a closed and normal BZ-cone, and if $-\mu_0 \in \rho(T)$ $(\mu_0 > 0)$ and $\mu_0 + T$ is positive, then there is a positive number $\eta > 0$ such that $\sigma(T) \subset \{\lambda: \operatorname{Re} \lambda > -\mu_0 + \eta\}$.

In section 13, the relationship between monotone completeness and order completeness are investigated. A partially ordered linear space is monotone-complete [sequentially monotone-complete] if each bounded non-decreasing transfinite sequence [sequence] has a supremum. Each monotone-complete vector lattice is order-complete. A partially ordered linear space E with a positive cone K is called a Z-space if $K - K = E$ and, whenever $0 \leq y \leq x_1 + x_2$ and $x_i \in K$, there are y_1 and y_2 such that $y = y_1 + y_2$ and $0 \leq y_i \leq x_i$ $(i = 1, 2)$. It is well-known that each vector lattice is a Z-space. The author proves that each monotone-complete Z-space is an order-complete vector lattice. Finally embeddings of Z-spaces in vector lattices are discussed.

In section 14, the relationship between order completeness and topological completeness is considered. Each monotone-complete and almost Archimedian (i.e., $-\xi y \leq x \leq \xi y$ for all $\xi > 0$ implies $x = 0$) partially ordered linear space with an order-unit is complete under the order topology. Let E be a partially ordered linear space with a metrizable locally convex topology which is stronger than the order topology and with a strict BZ-cone. If in addition each monotone and topologically bounded sequence has a supremum, then E is topologically complete.

Section 15 is devoted to the notion of quasi-interior points of a positive cone, which the author found to be useful in studying the spectra of positive operators [see Pacific J. Math. **10** (1960), 1009–1019; MR **22** #5893]. In a partially ordered linear topological space, a point x is called a q.i. (quasi-interior) point of the positive cone if the set $\{y: 0 \leq y \leq x\}$ is total. If E is a vector lattice a point $x \geq 0$ is called a weak order-unit if $x \wedge |y| = 0$ implies $y = 0$. If a vector lattice E has a locally convex topology with respect to which the positive cone K is normal and the map $x \to |x|$ is continuous, then each q.i. point of K is a weak order-unit. There are several other assertions concerning weak order units and q.i. points. In a partially ordered locally convex space E with a positive cone K, let K_q denote the set of all q.i. points of K; then K_q is either empty or dense in K. If E is separable and metrizable and K is complete and total, then K_q is not empty.

The final section of the present paper contains further comments (with additional theorems) on the material of the earlier two papers. The errors which are pointed out in the review of the first paper are corrected. In particular, Theorem 4.5 now reads: if E is a sequentially monotone complete partially ordered linear space, the order topology is tonnelé. *I. Namioka* (Ithaca, N.Y.)

11266:

Fréchet, Maurice. Sur une nouvelle définition des semi-espaces de Banach. C. R. Acad. Sci. Paris **251** (1960), 2629–2630.

The new definition is obtained from the original one [same C. R., 1258–1260; MR **22** #8312b] by adding the axioms: (1) $\|\xi - \eta\| \leq \|\xi - \zeta\| + \|\zeta - \eta\|$; (2) If $\xi - \xi_1 = \theta$, then $\xi = \xi_1$. *C. W. Kohls* (Rochester, N.Y.)

11267:

Chi, Guan-fu. Some remarks on linear functional sequences. Ann. Polon. Math. **8** (1960), 259–263.

The author considers the following problem: Let f_n be a sequence of linear functionals on a Banach space X such that $\lim_{n\to\infty} \|f_n\| = \infty$. Is it implied that $\lim_{n\to\infty} |f_n(x)| = \infty$ for some $x \in X$? If $\|f_n\|$ denotes the ordinary norm, then the answer is affirmative if and only if X is of dimension one. Next the author gives some conditions under which the above statement is true (in a somewhat more general form).

The following is a consequence of a theorem stated in the paper: There is a continuous non-decreasing real-valued function having an infinite right-hand derivative at each point of a dense subset of an interval.

P. Saworotnow (Washington, D.C.)

11268:

Garkavi, A. L. On spaces of linear operators. Dokl. Akad. Nauk SSSR **132** (1960), 497–500 (Russian); translated as Soviet Math. Dokl. **1**, 575–578.

Let X and Y be Banach spaces and let B be the space of all linear continuous functions from X into Y, with the usual norm. The author proves that if Y is the conjugate

of a Banach space $*Y$, then B is the conjugate of a space $*E$ which is the norm-closed linear hull in B^* of the set Γ of all those γ of the form $\gamma(F)=[F(x)](\phi)$ for all F in B, where $x \in U_X$, the unit ball of X, and $\phi \in *Y$.

The proof begins by noting that E, the space of all bounded functions from U_X into Y, is a conjugate space [see Day, *Normed linear spaces*, Springer, Berlin, 1958; MR **20** #1187; p. 31, (11b)]. The proof then uses the Kreĭn-Šmul'yan result [Ann. of Math. (2) **41** (1940), 556–583; MR **1**, 335] that it suffices to prove the unit ball U_B closed in the $\sigma(E, \Gamma)$ topology of E.

M. M. Day (Urbana, Ill.)

11269a:

Chang, Shih-Hsun. A generalization of Bunyakovskiĭ's inequality with applications to the theory of integral equations and Hilbert spaces. Acta Math. Sinica **7** (1957), 200–228. (Chinese. English summary)

11269b:

Chang, Shih-Hsun. A generalization of a formula of Landsberg and a further generalization of an inequality of Bunyakovskiĭ. Acta Math. Sinica **7** (1957), 229–234. (Chinese. English summary)

The author proves a generalization of the Schwarz-Bunyakovskiĭ inequality:

$$|\det(Kf_i, g_j)|^2 \leq \det(Kf_i, f_j)\det(Kg_i, g_j) \qquad (i, j = 1, 2, \cdots, n),$$

where f_i, g_j are elements of a Hilbert space, (,) denotes the inner product, and K is a bounded positive operator of the Hilbert space. Then he applies the inequality to the Hilbert space $L_2(a, b)$ and non-negative hermitian kernels $K(x, y)$ to obtain several inequalities relating the characteristic values of the kernels. These inequalities are too involved to be quoted.

When $n=1$ (and K is the identity operator), the above inequality becomes the Schwarz-Bunyakovskiĭ inequality. In this spirit, the author generalizes, in the second paper, a formula of Landsberg [Math. Ann. **69** (1910), 227–265]:

$$\frac{1}{n!}\int_a^b \cdots \int_a^b \det f_i(x_j)\det g_i(x_j)\, dv_x^{(n)} = \det \int_a^b f_i(x)g_j(x)\, dx.$$

Ti Yen (E. Lansing, Mich.)

11270:

Džrbašyan, M. M.; Martirosyan, R. M. On the general theory of biorthogonal kernels. Dokl. Akad. Nauk SSSR **132** (1960), 994–997 (Russian); translated as Soviet Math. Dokl. **1**, 683–687.

This continues the investigations of the first author [same Dokl. **128** (1959), 456–459; MR **21** #7407] on isometric and quasi-isometric transformations on $H_1 = L_{\sigma_1}^2(a_1, b_1)$ to $H_2 = L_{\sigma_2}^2(a_2, b_2)$. Call K and K_* conjugate on H if

$$\int_{a_1}^{b_1} \overline{K(\xi, x)}K_*(\eta, x)\, d\sigma_1(x) = \int_{a_2}^{b_2} e_\xi(x)e_\eta(x)\, d\sigma_2(x) \qquad (\xi, \eta \in (a_2, b_2)),$$

where $e_\xi \operatorname{sgn} \xi$ is the characteristic function of $[0, \xi)$ or $[\xi, 0)$, and call K complete if $\int_{a_1}^{b_1} K(\xi, x)f(x)d\sigma_1(x) = 0$ for all $\xi \in (a_2, b_2)$ implies that $f=0$; if such a K_* exists, K is called a B kernel, and is called a Bessel kernel if for any $f \in H_1$ there is a $g \in H_2$ such that

$$(1) \qquad \int_{a_1}^{b_1} f(x)\overline{K_*(\eta, x)}\, d\sigma_1(x) = \int_{a_2}^{b_2} g(\xi)e_\eta(\xi)\, d\sigma_2(\xi).$$

If $\bar{K}$ is conjugate to K, then it is a Bessel kernel and the relation $f \to g$ is an isometry; for K to be a Bessel kernel, it is necessary and sufficient that there be a bounded linear operator A on H_1 to itself such that $AK(\xi, x)$ is the kernel of an isometry, or alternatively, that there is a positive definite bounded hermitian operator T such that $K_* = TK$.

In order that for each $g \in H_2$ there is an $f \in H_1$ such that (1) holds, it is necessary and sufficient that $K = CK'$ where C is bounded and K' is the kernel of an isometric operator.

J. L. B. Cooper (Cardiff)

11271:

Džrbašyan, M. M.; Martirosyan, R. M. The problem of moments and the biorthogonalization of kernels. Dokl. Akad. Nauk SSSR **132** (1960), 1250–1253 (Russian); translated as Soviet Math. Dokl. **1**, 756–759.

Continuing the discussion of the paper reviewed above, conditions are given for kernels to admit conjugate kernels in terms of their closeness to isometric kernels in a certain (fairly complicated) sense. This gives a continuous analogue of results on biorthogonal sequences and on completeness of systems close to orthonormal complete systems of elements of a Hilbert space.

J. L. B. Cooper (Cardiff)

11272:

Louhivaara, Ilppo Simo. Über verschiedene Metriken in linearen Räumen. Ann. Acad. Sci. Fenn. Ser. A I No. 282 (1960), 22 pp.

In Hilbert space R let $H(x, y)$ represent the usual inner product, and let $B(x, y)$ represent a bounded $(|B(x, y)| \leq \alpha\sqrt{(H(x)H(y))}$, where $H(x) = H(x, x))$ bilinear form. The author deals with the following problem: Let Lx be a bounded $(|Lx| \leq \gamma\sqrt{H(x)})$ linear functional. When do there exist z^* $[z_*]$ in R such that $Lx = B(x, z^*)$ $[\overline{B(z_*, x)}]$? Littman [Comm. Pure Appl. Math. **11** (1958), 145–151; MR **20** #3382] gave a sufficient condition. Browder [Rend. Circ. Mat. Palermo (2) **7** (1958), 303–308; MR **21** #7365] derived a result which the author shows implies Littman's. These conclusions are drawn together for the discussion of B-normals. (If U is a closed linear subspace of R, $a \in R$, $a-p^*$ $[a-p_*]$ is called a right- [left-] sided B-normal provided p^* $[p_*] \in U$ and $B(u, a-p^*)$ $[B(a-p_*, u)] \equiv 0$, $u \in U$.)

To describe the theorem, certain definitions are needed: (1) U_0^* $[U_*^0] = \{u_0^*$ $[u_*^0]\,|\, \overline{B(u, u_0^*)}$ $[B(u_*^0, u)] \equiv 0, u \in U\}$; (2) $B_H(x, y) \equiv \frac{1}{2}(B(x, y) + \overline{B(y, x)})$; (3) $\{E_\lambda\}$, the spectral resolution of T defined by $H(Tx, y) = B_H(x, y)$ (note that $-\alpha \leq \lambda \leq \alpha$); (4) $U_\varepsilon^- = (\int_{-\alpha \leq \lambda < -\varepsilon} dE_\lambda)U$, $U_\varepsilon^+ = (\int_{\varepsilon < \lambda \leq \alpha} dE_\lambda)U$; (5) $u_{\varepsilon *}^-$ $[u_{\varepsilon *}^+] \in U_\varepsilon^-$ $[U_\varepsilon^+]$, unique elements depending on $u \in U$ and satisfying: $B(u, u_\varepsilon^-) = B(u_{\varepsilon *}^-, u_\varepsilon^-)$, all $u_\varepsilon^- \in U_\varepsilon^-$, and $B(u, u_\varepsilon^+) = B(u_{\varepsilon *}^+, u_\varepsilon^+)$, all $u_\varepsilon^+ \in U_\varepsilon^+$; (6) $u_*^\varepsilon = u - u_{\varepsilon *}^- - u_{\varepsilon *}^+$; (7) $U_*^\varepsilon = \{u_*^\varepsilon\}$, a closed linear subspace of U. Theorem: If a right-sided B-normal to U exists, then

$B(a_*^0, a)=0$ for all u_*^0 in U_*^0. If there is an $\varepsilon>0$ such that $B(u_*^\varepsilon, a)=0$ for all u_*^ε in U_*^ε, then a right-sided B-normal to U exists. The right-sided B-normal is unique modulo U_0^*.

Sharpenings of the preceding are derived for the case where B is Hermitian on U ($B(u_1, u_2)=\overline{B(u_2, u_1)}$, where $u_1, u_2 \in U$). *B. R. Gelbaum* (Princeton, N.J.)

11273:

Goodner, D. B. Separable spaces with the extension property. J. London Math. Soc. **35** (1960), 239–240.

It is shown that the only separable normed spaces having the Hahn-Banach extension property are finite-dimensional, with $\|(x_1, \cdots, x_n)\|=\operatorname{Max}_i |x_i|$. A topological condition on a space H is shown to imply that the space of bounded, continuous, real-valued functions on H has the Hahn-Banach property. {However, an extension of the author's reasoning shows that such spaces H are, in fact, countable and discrete.}

J. L. Kelley (Berkeley, Calif.)

11274:

Buck, R. C. Zero sets for continuous functions. Proc. Amer. Math. Soc. **11** (1960), 630–633.

Let D be a closed set in m-dimensional space and let V be a linear subspace of the space $C[D]$ of real-valued continuous functions defined on D, and let V be of dimension N. The author shows (Theorem 1) that there exist non-empty sets $D_1, \cdots, D_N$, relatively open in D, and a constant B, such that for any $f \in V$ and $p \in D$, and any choice of $p_i \in D_i$, we have

$$|f(p)| \leqq B\{|f(p_1)|+\cdots+|f(p_N)|\}.$$

In particular, if g and h are in V, and if they agree at some point in each of the sets $D_1, \cdots, D_N$, then they agree throughout D: thus these sets are "uniqueness domains" for V. The author shows also (Theorem 2) that if $D_1, \cdots, D_N$ is any collection of open disjoint non-empty sets of D, then there is a sub-space V of dimension N in $C[D]$ such that if $f \in V$, and if the set of zeros of f touches each of the sets D_i, then $f\equiv 0$.

H. P. Mulholland (Exeter)

11275:

Glicksberg, Irving. Weak compactness and separate continuity. Pacific J. Math. **11** (1961), 205–214.

Let X be a locally compact space, $C(X)$ all bounded continuous functions on X, $C_0(X)$ the functions in $C(X)$ that vanish at infinity, and $C_0(X)^*$ all finite complex regular Borel measures on X. Regard $C(X)$ as a subspace of $C_0(X)^{**}$. (1.1) If K is a bounded subset of $C(X)$, then K is compact in the topology of pointwise convergence on X if and only if it is compact in the weak-* topology of $C_0(X)^{**}$. (1.2) If f is a bounded separately continuous complex function on $X\times Y$, then $y\to\int f(x, y)\mu(dx)$ is continuous for $\mu \in C_0(X)^*$.

The author uses (1.2) to show that $\iint f(x, y)\mu(dx)\nu(dy)=\iint f(x, y)\nu(dy)\mu(dx)$ holds for $\mu \in C_0(X)^*$, $\nu \in C_0(Y)^*$, and a bounded separately continuous complex function on $X\times Y$. In §4, he uses (1.1) and (1.2) to extend where possible the results in his paper [same J. **9** (1959), 51–67; MR **21** #7405] to semigroups in which the operation is separately continuous. He also gives theorems involving the convergence of $(1/N)\sum_{n=1}^{N}\mu^{n}*f$, where μ is a normalized non-negative regular measure on a locally compact Abelian group and f is a weakly almost periodic function.

K. A. Ross (Seattle, Wash.)

11276:

Newman, D. J. The nonexistence of projections from L^1 to H^1. Proc. Amer. Math. Soc. **12** (1961), 98–99.

Let L^1 be the Banach space of all Lebesgue-integrable functions on the unit circle, and let H^1 consist of all f in L^1 whose Fourier coefficients $(2\pi)^{-1}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta$ are zero for all negative n. Clearly H^1 is a closed subspace of L^1. The author proves that there exists no bounded linear projection of L^1 onto H^1; in other words, L^1 is not a direct sum of H^1 and some other closed subspace.

W. Rudin (Madison, Wis.)

11277:

Havin, V. P. On the space of bounded regular functions. Dokl. Akad. Nauk SSSR **131** (1960), 40–43 (Russian); translated as Soviet Math. Dokl. **1**, 202–204.

Let F be a compact set in the plane, and G its complement. $A(G)$ is the space of functions holomorphic on G, and $A(F)$ the space of those holomorphic on F; each $f \in A(F)$ is in fact holomorphic on some neighborhood of F, and two functions that agree on such a neighborhood are identified. Both spaces have natural topologies and $A(G)$ becomes the dual space of $A(F)$ under the pairing $\langle g, f\rangle=(2\pi i)^{-1}\int_\Gamma g(t)f(t)dt$, $g \in A(G)$, $f \in A(F)$. [See G. Köthe, J. Reine Angew. Math. **191** (1953), 30–49; MR **15**, 132.] Let $B(G)$ be the subspace of $A(G)$ consisting of functions g that are bounded on G. The author obtains a norm topology on $A(F)$ for which $B(G)$ becomes the full dual space; he then investigates ways in which this can be used to study approximation problems in $A(F)$. [See also Havinson, Mat. Sb. (N.S.) **36 (78)** (1955), 445–478; MR **17**, 247.]

R. C. Buck (Princeton, N.J.)

11278:

Kilpi, Yrjö. Über das Regularitätsgebiet symmetrischer Transformationen im Hilbertschen Raum. Ann. Acad. Sci. Fenn. Ser. A I No. 288 (1960), 20 pp.

If H is a closed symmetric transformation in a Hilbert space $\mathfrak{H}$, the largest number $\alpha(a)$ such that $|Hf-af|\geq \alpha(a)|f|$, for every f in the domain of H, is of the form

$$\alpha(a)^2 = \alpha(\operatorname{Re} a)^2+(\operatorname{Im} a)^2.$$

A number a is said to be regular if $\alpha(a)>0$. If a is a real regular point such that $a_1=a-\alpha(a)$ and $a_2=a+\alpha(a)$ are also regular, there are non-zero elements $\varphi(a_1)$ and $\varphi(a_2)$ of $\mathfrak{H}$ with $\varphi(a_1)$ orthogonal to the range of $H-a_1$ and in the range of $H-a_2$, whereas $\varphi(a_2)$ is orthogonal to the range of $H-a_2$ and in the range of $H-a_1$. This fact is related to a self-adjoint extension of H in $\mathfrak{H}$, the domain of H being assumed dense. The multiplicity of eigenvalues of this extension cannot exceed the (equal) deficiencies of H, assumed finite. The theorems are illustrated for the transformation $-id/dx$ in the space $L^2(0, 1)$.

L. de Branges (Bryn Mawr, Pa.)

11279:

Olagunju, A. A note on closed operators. Proc. Cambridge Philos. Soc. **57** (1961), 426.

Author's summary: "If E is a Banach space, A is a

closed operator from E into itself and P is a projection of rank 1 on E, is the product PA necessarily closed? In this note, we show that the answer is negative. . . ."

D. A. Edwards (Newcastle upon Tyne)

11280:

Harazov, D. F. The convergence of the method of steepest descent for operators depending quadratically on a parameter. Bul. Inst. Politehn. Iași (N.S.) **5 (9)** (1959), no. 1–2, 35–40. (Russian. English and Romanian summaries)

Let A_1 and A_2 be compact linear operators in a separable real Hilbert space X, and let H be a bounded positive definite symmetric operator. Suppose that $(HA_ix, y) = (x, HA_iy)$ $(x, y \in X;\ i=1, 2)$, $(Hx, x) > 0$, $(HA_2x, x) > 0$ $(x \neq 0)$. The author considers equations of the form

$$(1) \qquad x - \lambda A_1x - \lambda^2A_2x = y.$$

He forms a new scalar product on $X \times X$, defined by $[\{x_1, y_1\}, \{x_2, y_2\}] = (Hx_1, x_2) + (HA_2y_1, y_2)$, and writes $A\{x, y\} = \{A_1x + A_2y, x\}$. It then turns out that A is symmetric in the (in general incomplete) inner-product space $G = X \times X$. Applying Kantorovič's method of steepest descent [L. V. Kantorovič, Uspehi Mat. Nauk **3** (1948), no. 6 (28), 89–185; MR **10**, 380] to the operator A, the author obtains iterative methods of finding the smallest characteristic value λ of the equation $x - \lambda A_1x - \lambda^2A_2x = 0$, and of solving the non-homogeneous equation (1) for a certain range of values of λ.

F. Smithies (Cambridge, England)

11281:

Bonic, Robert A. Symmetry in group algebras of discrete groups. Pacific J. Math. **11** (1961), 73–94.

Let A be a Banach algebra with an isometric involution $*$. The involution is said to be hermitian if $x = x^*$ implies that x has a real spectrum. The algebra A is symmetric if for any $y \in A$, the spectrum of y^*y is real and non-negative. This paper studies these properties for the group algebra $l_1(G)$ of a discrete group G.

The principal results are the following: (3.4) If G is a discrete abelian group and H a discrete group such that $l_1(G)$ is symmetric [has an hermitian involution], then $l_1(G \times H)$ is symmetric [has an hermitian involution]. (3.7) If K is a discrete abelian group, C is any finite group of automorphisms of K, and G is the corresponding semi-direct product of K and C, then the involution of $l_1(G)$ is hermitian. (3.9) If K is a discrete abelian group, C is a two-element group of automorphisms of K, and G is the semi-direct product, then $l_1(G)$ is symmetric. It is not known whether $l_1(G)$ is symmetric if C is an arbitrary finite group of automorphisms of K. (3.10) Let G and H be discrete, $H \subset G$. If $l_1(G)$ is symmetric [has an hermitian involution], then $l_1(H)$ is symmetric [has an hermitian involution]. Examples for which the involution of $l_1(G)$ is not hermitian are given.

An interesting related result is (5.8), which gives a sufficient condition for a countable group to fail to have an invariant mean.

K. A. Ross (Seattle, Wash.)

11282:

Comfort, W. W. The Šilov boundary induced by a certain Banach algebra. Trans. Amer. Math. Soc. **98** (1961), 501–517.

Let G be a commutative semigroup, and consider the Banach algebra $l_1(G)$ of all complex-valued functions α on G for which $\sum_{z \in G} |\alpha(z)| < \infty$. A semicharacter is a nonzero homomorphism χ of G into $\{w\colon w$ is a complex number and $|w| \leq 1\}$. If $\hat{G}$ denotes the set of semicharacters of G, then there is a 1-1 correspondence between $\hat{G}$ and the maximal ideal space $\mathfrak{M}$ of $l_1(G)$. The Gel'fand topology of $\mathfrak{M}$ is transferred to $\hat{G}$ by this correspondence. For $\alpha \in l_1(G)$ and $\chi \in \hat{G}$, define $\hat{\alpha}(\chi) = \sum_{z \in G} \alpha(z)\chi(z)$. Then $\{\hat{\alpha}\colon \alpha \in l_1(G)\}$ is a separating algebra of continuous functions vanishing at infinity on the locally compact Hausdorff space $\hat{G}$. Let ∂ denote the Šilov boundary induced in $\hat{G}$ by $\{\hat{\alpha}\colon \alpha \in l_1(G)\}$. Let $B = \{\chi \in \hat{G}\colon |\chi| = 1\}$ and $\Gamma = \{\chi \in \hat{G}\colon |\chi| = 0 \text{ or } 1\}$. For $\chi \in \hat{G}$, let $S(\chi) = \{z \in G\colon \chi(z) \neq 0\}$.

The principal results are the following. The inclusions $B \subset \partial \subset \Gamma$ and $B\partial \subset \partial$ hold. Theorem 6.4: If $\hat{G}$ separates points of G, the following are equivalent: (a) G is a union of groups; (b) $\partial = \hat{G}$; (c) $\Gamma = \hat{G}$. Theorem 8.7: For a finite subset F of G, let $\hat{G}_F$ denote all $\chi \in \Gamma$ such that $y \in F$ implies either (1) there exist $x_1, x_2 \in S(\chi)$ such that $x_1y = x_2y$ and $\chi(x_1) \neq \chi(x_2)$, or (2) there exist $x_1, x_2 \in S(\chi)$ and $z \in G - S(\chi)$ such that $x_1yz = x_2y$. Then $\chi \in \partial$ if and only if $\chi \in (\hat{G}_F)^-$ whenever F is a finite subset of G disjoint from $S(\chi)$.

A number of examples are given, e.g., all the inclusions $B \subset \partial \subset \Gamma \subset \hat{G}$ can be proper for a single semigroup G.

K. A. Ross (Seattle, Wash.)

11283:

Olubummo, A. On the existence of an absolutely minimal norm in a Banach algebra. Proc. Amer. Math. Soc. **11** (1960), 718–722.

The norm $\|\cdot\|$ in a Banach algebra A is said to be absolutely minimal if, whenever $\|\cdot\|_1$ is a norm with respect to which A is a normed algebra, $\|a\| \leq \|a\|_1$ $(a \in A)$. By a theorem of Kaplansky [Duke Math. J. **16** (1949), 399–418; MR **11**, 115], the uniform norm in an algebra $C(E)$ of continuous functions is absolutely minimal. It is proved here that if X is a Banach space of dimension greater than one and A is an algebra of operators on X that contains all operators of finite rank, then there does not exist an absolutely minimal norm with respect to which A is a Banach algebra. An example is also given, using quaternion-valued functions, of a non-commutative algebra with an absolutely minimal norm.

F. F. Bonsall (Newcastle upon Tyne)

11284:

Katznelson, Yitzhak. Calcul symbolique et endomorphismes dans quelques algèbres de Banach. Séminaire P. Lelong, 1958/59, exp. 6, 6 pp. Faculté des Sciences de Paris, 1959.

Summary of the author's doctoral thesis [Ann. Sci. École Norm. Sup. (3) **76** (1959), 83–123; MR **22** #165].

H. Mirkil (Hanover, N.H.)

11285:

Göhde, Dietrich. Über Fixpunktsätze und die Theorie des Abbildungsgrades in Funktionalräumen. Math. Nachr. **20** (1959), 356–371.

If X is the residual space of a closed sphere of an infinite-dimensional Banach space after N disjunct open spheres U_i are removed, then a completely continuous self-map, F, of X has a fixed point. (The related case X

and $\{\bar{U}_i\}$ absolute retracts is a result of the reviewer [Rev. Math. Pures Appl. **2** (1957), 371–374; MR **20** #1297].)

Other results are connected with invariance under homotopy. For instance, in the Hilbert space, H, let $f_i = 1 + F_i$, $i = 1, 2$, be self-maps of the unit sphere, S, where F_i is completely continuous and 1 is the identity map. Then f_1 can be deformed to f_2 if the Leray-Schauder degree is 0. If f_1 is given, then a completely continuous F_2 exists such that f_2 is the extension of f_1 to a self-map on the unit ball of H to S. *D. G. Bourgin* (Urbana, Ill.)

11286:

Nevanlinna, Rolf. Über die Methode der sukzessiven Approximationen. Ann. Acad. Sci. Fenn. Ser. A I No. 291 (1960), 10 pp.

If $y = y(x)$ is a single-valued function on the sphere $|x| < r_0 \leq \infty$ of a linear normed complete space X to X, then the Picard successive-approximation method yields an inverse for $y(x)$ on $|y| < \rho_0$, provided

$$\theta(r) = \sup_{x \leq r} \limsup_{\Delta x \to 0} |y(x+\Delta x) - y(x) - \Delta x|/|\Delta x| < 1,$$

for $0 < r < r_0$ and $\rho_0 = r_0 - \int_0^{r_0} \theta(r)dr$, where the integral is the upper Riemann integral. This by-passes the usual differentiability assumption on y [cf. remark in abstract MR **21** #2932]. The proof is based on the lemma: If for $|x| < r_0$ we have $\lim \sup_{\Delta x \to 0} |f(x+\Delta x) - f(x)|/|\Delta x| \leq M(x)$, then for $|a|, |b| < r_0$: $|f(b) - f(a)| \leq \int_a^b M(x)|dx|$, where the upper Riemann integral is taken along the line joining a to b. *T. H. Hildebrandt* (Ann Arbor, Mich.)

CALCULUS OF VARIATIONS

11287a:

Berruti Onesti, Natalia. Sopra le estremali relative ad integrali curvilinei dello spazio in forma parametrica. Ann. Mat. Pura Appl. (4) **52** (1960), 79–106.

11287b:

Berruti Onesti, Natalia. Sopra le estremaloidi relative ad integrali curvilinei dello spazio in forma parametrica. Ann. Mat. Pura Appl. (4) **52** (1960), 219–246.

The papers consider integrals in parametric form involving derivatives up to the second or the third order of the functions representing the curve in three-space. In the first paper, the minimizing curve is assumed to be of class C'' (C''' in case third-order derivatives appear). Extensions of the Euler equations are derived, and theorems are given on the existence of derivatives of order higher than the second (or third), when the minimizing curve is represented in terms of the arc length.

In the second paper, under additional restrictions on the integrand F, it is shown that any minimizing curve must satisfy an extension of the du Bois-Reymond equations. Since the case when third-order derivatives may appear in the integrand includes the other, the proof is given here in detail only for this more complicated case. It is preceded by indications of a simpler proof possible for the case when derivatives of order higher than the second do not appear.

The derivatives of second and third order enter the integrand only in the combinations $u_2 = z'x'' - z''x'$, $v_2 = y'z'' - y''z'$, $w_2 = z'x'' - z''x'$, $u_3 = x'y''' - x'''y'$, etc. However, conditions on F to ensure that the integral is independent of choice of parameter are not completely specified. Arc length is taken as parameter in the theorems, but not on the varied curves in the proof of the Euler equations. In the examples considered, the integrand functions F are given in the special forms which they take whenever arc length is the parameter, and the hypotheses of the theorems seem to be meant to apply to those special forms.

L. M. Graves (Chicago, Ill.)

GEOMETRY

See also 10885, 10911, 11044, 11045, 11046, 11251, 11354.

11288:

Efimow, N. W. ★Höhere Geometrie. Hochschulbücher für Mathematik, Bd. 51. VEB Deutscher Verlag der Wissenschaften, Berlin, 1960. viii+556 pp. DM 32.00.

In the course of translation from Russian, the third edition [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953; MR **16**, 395] has been modified by the author by the addition of a chapter on Minkowski space and the foundations of the special theory of relativity.

11289:

Johnson, Roger A. ★Advanced Euclidean geometry: An elementary treatise on the geometry of the triangle and the circle. (Formerly titled: Modern geometry.) Under the editorship of John Wesley Young. Dover Publications, Inc., New York, 1960. xiii+319 pp. $1.65.

Unaltered republication of the 1st edition [Houghton Mifflin, Boston, 1929]. Contents (in brief): Similar figures; coaxal circles, inversion; general geometry of triangles, polygons, circles; theorems of Miguel, Ceva, Menelaus; inscribed and escribed circles; nine-point circle; symmedian and other notable points; perspective triangles; pedal triangles and circles; Brocard configuration; equibrocardal triangles.

11290:

Fladt, Kuno. Über die Punktrechnung Hermann Grassmanns d. Ä. Jber. Deutsch. Math. Verein. **62**, Abt. 1, 99–129 (1960).

Der Verfasser beschäftigt sich in der vorliegenden Arbeit mit der bisher noch unbearbeiteten Punktrechnung von Hermann Grassmann d. Ä. Zuerst wird das Axiomensystem der Punktrechnung zusammengestellt. Dabei wird eine Verbindung zwischen Punkten und reellen (später auch komplexen) Zahlen hergestellt. Die Kommutativität der Multiplikation wird nirgends benutzt, und dementsprechend wird die Punktrechnung über einem Schiefkörper definiert. In der Arbeit wird auch darauf hingewiesen, dass die wirkliche Punktrechnung nicht zu den inhomogenen Koordinaten führt – wie dies z.B. im Buche von O. Veblen und J. W. Young [*Projective geometry*, Ginn, Boston, Vol. I, 1910, Vol. II, 1918] der Fall ist – sondern zu den projektiven Koordinaten, so dass die Punktrechnung nicht nur zu der affinen Gruppe, sondern durchaus zu der projektiven Gruppe gehört.

Beim Ausbau der Punktrechnung können wesentliche

Vereinfachungen erzielt werden, wenn von den Grassmannschen äusseren oder alternierenden Produkten der Koordinateneinheiten Gebrauch gemacht wird. Mittels dieser äusseren Produkte definiert der Verfasser in der Ebene den "Stab" (Strecke), im Raum das "Blatt" und den "Block" sowie Summen und Produkte derselben, ferner die Ordnung von Punkten und projektiv gleiche Strecken. Im abschliessenden Teil der Arbeit werden sog. projektive Vektoren definiert, und es wird auf die Summation und auf das äussere Produkt derselben eingegangen. *L. Gyarmathi* (Debrecen)

11291:

Goormaghtigh, R. Généralisation d'un théorème de Zeeman. Mathesis **68** (1959), 356–361.

The final form of the theorem is: Δ_1, Δ_2, Δ_3, Δ_4 are four lines in a plane; H_3, H_4 are respectively the orthocentres of $\Delta_1\Delta_2\Delta_4$, $\Delta_1\Delta_2\Delta_3$; M_4 is arbitrary and its barycentric coordinates relative to $\Delta_3\Delta_2\Delta_1$ are α, β, γ; M_3 is the point whose barycentric coordinates relative to $\Delta_4\Delta_1\Delta_2$ are also α, β, γ. The line through $\Delta_1\Delta_2$ parallel to Δ_i $(i=3, 4)$ meets M_iH_i in Q_i; then $M_3Q_3/Q_3H_3 = M_4Q_4/Q_4H_4$. In the course of the proof various aspects of Zeeman's theorem and N. G. de Bruijn's generalization are discussed. *T. G. Room* (Sydney)

11292:

Mandan, Sahib Ram. On four intersecting spheres. J. Indian Math. Soc. (N.S.) **23** (1959), 151–167 (1961).

From the author's summary: "The radical tetrahedron of four intersecting spheres coincides with a diagonal tetrahedron of the desmic system of intersection of those spheres. The planes of perspectivity of the eight pairs of complementary tetrahedra of intersection of four intersecting spheres form two tetrahedra desmic with their radical tetrahedron. The diagonal tetrahedra of a desmic system of intersection of four intersecting spheres form the other desmic system of intersection of those spheres. In the end corresponding results for four mutually orthogonal and real spheres are deduced. Finally umbilical projection is suggested as a process to get these results rather quickly."

11293:

Wise, M. E. On the radii of five packed spheres in mutual contact. Philips Res. Rep. **15** (1960), 101–106. (French and German summaries)

Let $x_1, \cdots, x_5$ denote the reciprocals of the radii of five spheres in mutual contact (with a minus sign if the contact is internal). The author uses inversion to prove that

$$(x_1+\cdots+x_5)^2-3(x_1^2+\cdots+x_5^2) = 0.$$

This result was anticipated by Frederick Soddy, whose poem "The kiss precise" states it thus:

A fifth sphere in the kissing shares.
Yet, signs and zero as before,
For each to kiss the other four
The square of the sum of all five bends
Is thrice the sum of their squares.

[Nature **137** (1936), 1021; see also Coxeter, Scripta Math. **18** (1952), 113–121; MR **14**, 494; p. 116.] *H. S. M. Coxeter* (Toronto, Ont.)

11294:

Skof, Fulvia. Costruzioni grafiche del piano osculatore ad una quartica gobba di prima specie col metodo delle corde equivalenti. Period. Mat. (4) **37** (1959), 156–174.

The author applies the method of "equivalent chords", due to U. Cassina [Ist. Lombardo Rend. (2) **68** (1935), 503–517], to the graphical construction of the osculating plane of a quartic of first species. The method is based on the application of two theorems (which are stated): the first serving when the quartic is the intersection of two cones (or cylinders) of second degree; the second in the general case, also when there are no real cones passing through the curve.

Then the author develops the following cases: (a) the quartic is the intersection of two cones, in central projection; (b) the quartic is the intersection of two quadrics, in the Monge projection (as quartic is taken the Viviani curve); (c) the quartic is the intersection of two ruled quadrics, in the Monge projection, each quadric being given by three generators of the same system. *D. Mazkewitsch* (Cincinnati, Ohio)

11295:

Jovičić, M. M. Eine neue Methode zum Zeichnen einer Schrägperspektive mittels einer Frontalperspektive. Elem. Math. **15** (1960), 127–130.

Der Verfasser gibt eine Methode zum Umzeichnen einer Perspektive, die die kollineare Beziehung zwischen zwei Perspektiven aus demselben Auge auf zwei verschiedene Bildebenen verwertet und den Zweck hat die Schwierigkeit unzugänglicher Fluchtpunkte zu umgehen. *M. Piazzolla-Beloch* (Ferrara)

11296:

Drs, Ladislav. Umzeichnen von Perspektiven bei ungleichgeneigten Bildebenen. Elem. Math. **15** (1960), 131–133.

Der Verfasser gibt ein graphisches Verfahren zum Umzeichnen von Perspektiven (aus verschiedenen Augen genommen) bei ungleichgeneigten Bildebenen. *M. Piazzolla-Beloch* (Ferrara)

11297:

Mandan, Sahib Ram. Projective tetrahedra in a 4-space. J. Sci. Engrg. Res. **3** (1959), 169–174.

The author describes, in projective 4-space, a set of eight points that can be divided, in eight ways, into two sets of four, forming "perspective" tetrahedra whose pairs of corresponding face-planes meet in four collinear points, while their pairs of corresponding vertices are joined by four lines having a transversal. The eight such transversals lie in a hyperplane and belong to a regulus. *H. S. M. Coxeter* (Portland, Ore.)

11298:

Vogler, H. Ein räumliches Analogon zur Aufgabe von Ottajano. Elem. Math. **15** (1960), 101–103.

Given in projective three-space a quadric Φ and n points $A_1, \cdots, A_n$, find a set of n points P_i on Φ such that P_iP_{i+1} passes through A_i $(i=1, \cdots, n;\ A_{n+1}=A_1)$. The problem is solved by considering the product of the n involutory projectivities of Φ into itself, in which the fixed elements are the points A_i and their polar planes. The various types of possible solutions are detailed. *T. G. Room* (Sydney)

11299:

Feld, J. M. An application of turns and slides to spherical geometry. Amer. Math. Monthly **66** (1959), 665–673.

Dans cette étude l'auteur commence par appliquer la théorie des quaternions d'Hamilton à l'établissement des relations qui existent entre trois types de transformations élémentaires des éléments linéaires tangents de la sphère unitaire (turns, slides, dilatations), cas particuliers d'une transformation plus générale (spherical whirl) dont les opérations sont celles d'un groupe à trois paramètres, et dont ils constituent des sous-groupes à un paramètre. Par application des résultats obtenus, il indique ensuite une méthode élégante pour démontrer les formules de Delambre liant les sinus et les cosinus des demi-côtés et des demi-angles d'un triangle sphérique, et des formules analogues relatives aux polygones sphériques.

P. Vincensini (Marseille)

11300:

Zobel, A. A condition calculus on an open variety. Rend. Mat. e Appl. (5) **19** (1960), 72–94.

The author's aim is to follow further Severi's idea of extending algebraic equivalence and virtual intersection to an "open variety", i.e., an algebraic variety W from which a subvariety S has been excluded (S in fact contains all the singularities of W). Because of the difficulties encountered in the more generalized formulation, he limits the discussion to varieties which are generic in irreducible systems, and succeeds in obtaining significant new results.

Of equal value with these results is the discussion of certain aspects of the concept of generality, in the course of which the author proposes a number of useful definitions.

T. G. Room (Sydney)

11301:

Guggenheimer, Heinrich. On the topology of the exceptional varieties in birational transformations. Ann. Mat. Pura Appl. (4) **52** (1960), 1–8. (Italian summary)

A birational transformation between two algebraic varieties V, V' is given by an algebraic subvariety W of $V\times V'$ (V, V', W having the same dimension n) such that the projections $W\to V$ and $W\to V'$ are analytical mappings of degree one. Then the exceptional varieties are the minimal algebraic subvarieties $M\subset V$, $N\subset W$, $M'\subset V'$, such that the analytical mappings $W-N\to V-M$ and $W-N\to V'-M'$ are one-to-one onto.

Here the relations between the homology properties of V, V', W, M, M', N are investigated in the classical case (i.e., in the complex field; the case of Cremona transformation, i.e., when V, V' are complex projective space, was already dealt with by the reviewer [same Ann. (4) **43** (1957), 1–23; MR **19**, 579], where the torsion coefficients were also considered).

Among other things it is shown that, if V and V' are nonsingular, then the Betti numbers of V, M, V', M' are connected by

$$b_r(V)-b_r(M) = b_r(V')-b_r(M') \quad (0\leq r\leq n).$$

Next the result is extended to singular varieties, under the assumption [as yet only proved in a special case: cf. B. Segre, ibid. **33** (1952), 5–48; Rend. Circ. Mat. Palermo (2) **1** (1952), 373–379; MR **14**, 683; **15**, 351] that every algebraic variety may be transformed into a nonsingular one by means of a finite number of dilatations.

B. Segre (Rome)

11302:

Coolidge, Julian Lowell. ★A treatise on algebraic plane curves. Dover Publications, Inc., New York, 1959. xxiv + 513 pp. Paperbound: $2.45.

Unaltered republication of the first edition [Oxford Univ. Press, 1931].

11303:

Godeaux, Lucien. Une involution appartenant à la surface intersection de quatre hyperquadriques de l'espace à six dimensions. Acad. Roy. Belg. Bull. Cl. Sci. (5) **46** (1960), 105–112.

Let I be a cubic involution in S_6 generated by a cyclic homography whose invariant loci are a plane σ and two lines r_1, r_2. There are three linear systems of quadrics belonging to I, one of dimension 9 with r_1, r_2 as base loci, and two of dimension 8 with σ, r_1, resp. σ, r_2, as base loci. F is the surface of intersection of two general quadrics of the first system with one of each of the others. F belongs to I, and contains eight united points, two each on r_1, r_2 and four in σ; the former are of the first kind (every point of the first neighbourhood being united) and the latter of the second kind (with only two united points in the first neighbourhood).

A model of the involution I on F is constructed as projective model F' of the sections of F by quadrics of the first system. F' is of order 16 in S_7, and is the intersection of two quadrics (one a cone with S_3 as vertex) with the cone projecting a Veronese surface from a line. F' is the bicanonical model of a regular surface of genera $p_g=p_a=3$, $p^{(1)}=5$. The images of the united points of the first kind are conics on F', those of the others are binodes B_3.

P. Du Val (London)

11304:

Godeaux, Lucien. Sur les surfaces de genres arithmétique et géométrique zéro dont le système bicanonique est irréductible. Acad. Roy. Belg. Bull. Cl. Sci. (5) **46** (1960), 743–747.

Dans deux communications antérieures [même Bull. **45** (1959), 362–372; **46** (1960), 47–52; MR **22** #7040, 7041], l'auteur a démontré (1ère communication) que si une surface F de genres $p_a=p_g=0$, $p^{(1)}\geqq 3$ possède un système bicanonique irréductible, elle admet deux systèmes linéaires $|\Gamma_1|$, $|\Gamma_2|$ tels que, $|C_i|$ étant le système i-canonique, on ait $|C_3|=|\Gamma_1+\Gamma_2|$, $|C_4|=|\Gamma_1+\Gamma_2'|=|\Gamma_2+\Gamma_1'|$, $|C_5|=|\Gamma_1'+\Gamma_2'|$, puis (2ème communication) que les systèmes $|\Gamma_1|$, $|\Gamma_2|$ sont réguliers et découpent des séries complètes sur une courbe bicanonique. Il complète ici ces résultats en examinant le cas où le système $|\Gamma_1|$ se réduit à une courbe isolée. Il démontre qu'alors le deuxième système $|\Gamma_2|$ est l'adjoint $|\Gamma_1'|$ à $|\Gamma_1|$, et retrouve ainsi une surface étudiée dans la 1ère des deux communications antérieures, représentant une involution du 2ème ordre privée de points unis possédant une seule courbe canonique.

P. Vincensini (Marseille)

11305:

Goormaghtigh, R. Sur les triangles de Poncelet. Mathèsis **69** (1960), 49–53.

L'auteur complète ici des résultats antérieurs [Mathèsis **53** (1939), 269–276; **58** (1949), 24–29; MR **1**, 155; **11**, 126],

qu'il a établis par l'application d'un principe basé sur l'emploi des coordonnées complexes dans le plan, et relatifs à divers lieux géométriques déduits de la considération d'un triangle variable (T) inscrit dans une circonférence Γ et circonscrit à une conique Σ. Il montre que la plupart des éléments remarquables (points ou droites) attachés à (T) décrivent ou enveloppent des coniques, les foyers de la conique inscrite dans (T) et concentrique à Γ décrivant une courbe de Cassini. Le cas où Σ est une parabole échappe à l'analyse générale; l'auteur en fait une étude directe et signale les principaux résultats qui s'y rattachent. *P. Vincensini* (Marseille)

11306:

Al-Dhahir, M. W. A note on the two-quadrangle theorem. Bull. Coll. Sci. Baghdad **4** (1959), 60–62. (Arabic summary)

The two-quadrangle theorem for the oppositely placed case [Veblen and Young, *Projective geometry*, *Vol. 1*, Ginn, Boston, 1938; p. 101] is proved analytically. Furthermore, Pappus' theorem is shown geometrically to be a specialization of the theorem. This reaffirms the contrast between the two cases of the two-quadrangle theorem, which in its similarly placed case [op. cit., p. 47] follows from Desargues' theorem alone.

R. Artzy (Chapel Hill, N.C.)

11307:

Kelly, Paul; Straus, E. G. On the projective centres of convex curves. Canad. J. Math. **12** (1960), 568–581.

Recherche des homologies qui transforment en elles-mêmes une courbe plane convexe du plan projectif; on peut regarder ces transformations comme symétries du plan de Hilbert. L'auteur étudie d'abord les groupes continus de telles transformations; les trajectoires des points d'un sous-groupe ∞^1, représenté par e^{tA} où A est matrice 3×3, sont telles qu'une courbe convexe dotée d'un tel groupe est formée, soit de lignes brisées, soit de deux arcs des courbes $y=x^m$ et $y=-x^m$ $(0<m<1)$, soit de $y=x^m$ et de $x=0$, et droite de l'infini, soit enfin de $y=x\log x$ et de $x=0$. S'il existe une suite de centres projectifs ayant une limite sur la courbe C, ou C est conique et tout point non de C est centre, ou C est formé de deux arcs de coniques distinctes convenablement raccordées et les centres sont tous les points de la corde commune extérieurs à C, ou un arc de conique et sa corde et les centres sont les mêmes que précédemment; ceci se montre par recours au plan de Hilbert. S'il y a deux centres intérieurs, il y en a une infinité sur la droite qui le joint; aux extrémités de cette corde, C, si non conique, possède une courbure avec singularité de 2° espèce; même résultat si les points sont extérieurs; si la suite des centres est finie, le nombre en est impair avec un seul intérieur, les autres alignés sur une droite qui ne rencontre pas C; le groupe est alors celui des symétries du polygone régulier. Il existe des courbes convexes avec infinité de centres indépendants, et toute courbe coïncide avec une telle courbe sauf sur des segments de mesure linéaire aussi petite que l'on veut. Généralisations à des espaces de plus haute dimension. *B. d'Orgeval* (Dijon)

11308:

Lunelli, L.; Lunelli, M.; Sce, M. Calcolo di omografie cicliche di piani desarguesiani finiti. Rend. Mat. e Appl. (5) **18** (1959), 351–374.

On sait [J. Singer, 1938] qu'un plan projectif arguésien fini d'ordre q ($q+1$ points sur chaque droite) est doué d'homographies cycliques, c'est-à-dire d'homographies dont la période, q^2+q+1, est égale au nombre des points (des droites) du plan. Ayant fixé un point, P, et une homographie cyclique, ω, on peut associer à chaque point R un nombre entier, n_R, modulo q^2+q+1, tel que: $R=\omega^{n_R}(P)$; on dira qu'on a "ordonné" les points du plan. Les auteurs ont déterminé, à l'aide de la machine électronique à calculer de l'École Polytechnique de Milan: (1) les "dispositions" des points des plans arguésiens projectifs finis d'ordre 2, 3, 4, 5, 7, 9, 11 associées à certaines homographies cycliques (§ 3); (2) toutes les formes canoniques des homographies cycliques qui existent dans les plans des ordres susdits, à l'exception de l'ordre 9 (§ 4). Les auteurs donnent aussi des renseignements sur les programmes adoptés pour le calcul (§ 2); ils soulignent l'intérêt des homographies cycliques, et des "dispositions" des points d'un plan engendrées par elles, dans la représentation du fonctionnement de certaines systèmes automatiques (§ 1).

L. Lombardo-Radice (Rome)

11309:

Cossu, Aldo. Sulle ovali in un piano proiettivo sopra un corpo finito. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 342–344.

L'auteur donne une démonstration simple et élégante d'un théorème de B. Segre (1954): "chaque ovale d'un plan linéaire sur un corps de Galois GF(q), q impair, est une conique", et aussi de quelques autres résultats atteints par B. Segre dans le même ordre de questions.

L. Lombardo-Radice (Rome)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 10979, 11293, 11307.

11310:

Hadwiger, H.; Debrunner, M. ★Kombinatorische Geometrie in der Ebene. Monographies de "L'Enseignement Mathématique", No. 2. Institut de Mathématiques, Université, Genève, 1960. 122 pp. 20 fr. s.

This book is a substantially expanded version of the authors' paper, Enseignement Math. (2) **1** (1955), 56–89 [MR **17**, 887]. As in the original paper, the book is divided into a first part containing statements of theorems together with discussion and illustrations, and a second part giving the proof of the statements. For general comments on this excellent work, the reader is referred to the earlier review. We shall here mention briefly the new material covered.

The present volume contains statements and proofs of ninety-eight theorems as against only thirty-seven in the original paper. The first thirty-five theorems of the book are the same as those of the paper. The results of the original paper concerning the notion of diameter, however, have been included in a new section entitled "Überdeckungsprobleme". Among the new theorems included here are Sperner's Lemma, a special case of the "plank problem" stating that if a convex body can be covered

by two strips of width a and b, then it can be covered by a single strip of width $a+b$, and the fact that any convex body other than a parallelogram can be covered by the interiors of three translates of itself. The next section "Punktmengengeometrie und Konvexität" includes the "art gallery" theorem: If any three pictures in the gallery can be seen from some point then there is a point from which all pictures can be seen. The next section, well described by its title "Realisierung von Distanzen", contains ten theorems, among which the following is typical: If a line is covered by two closed subsets, one of the two contains points at all possible distances. On the other hand, an example shows that continuous curves uniformly close to a line need not have this property. In "Einfache Paradoxien bei Punktmengen", we learn that a bounded set can be congruent to a proper subset, but a compact set cannot; that an unbounded set can be the union of two disjoint sets congruent to it, but a bounded set cannot; that an unbounded set can be "shrunk" onto itself, and a bounded set can be shrunk without decreasing its diameter; that no compact convex set is the union of two disjoint congruent subsets, though there exists a bounded symmetric set containing its center of symmetry which is nevertheless the union of disjoint congruent subsets. The next section "Rein Kombinatorik, Streckenkomplexe" contains eleven theorems of graph theory which have little to do with "Geometrie in der Ebene" although they are certainly "kombinatorische". The final section "Weitere Sätze von Hellyschen Typ" contains first results of the form: If, in a family of sets of a certain class, every p-set subfamily contains q sets with a non-empty intersection, then the family can be partitioned into $N(p, q)$ subfamilies each with non-empty intersection. Helly's Theorem states that $N(3, 3)=1$. If the family consists of segments on a line, then $N(p, q)=p-q+1$. Next there is a collection of results concerning common transversals to a family of convex sets. The necessary condition that every three sets have a common transversal is shown to be sufficient as well in a number of special cases. The final results are some Helly-like theorems for infinite families. *D. Gale* (Providence, R.I.)

11311:

Rešetnyak, Yu. G. Integration over a convex polyhedron and some problems in the theory of linear inequalities. Dokl. Akad. Nauk SSSR **130** (1960), 981–983 (Russian); translated as Soviet Math. Dokl. **1**, 122–124.

It is shown how to find exactly the integral of $\exp(a_1x_1+\cdots+a_nx_n)$ over a convex polyhedron, the latter being defined by a system of linear inequalities. The proof makes use of known properties of integrals of the form $\int f(s)\exp(xs)ds$, where $f(s)$ is a simple rational function, and the integration is along a line parallel to the imaginary axis. A rigorous justification for changing the order of certain integrations is not given. It is indicated how the technique can be used in the determination of the dimension of the polyhedron, and of its boundaries. It is also shown how one could find a supporting plane to the polyhedron.

T. E. Hull (Vancouver, B.C.)

11312:

Knothe, Herbert. On polyhedra in spaces of constant curvature. Michigan Math. J. **7** (1960), 251–255.

The author obtains the expression

$$\tfrac{4}{3}\pi^2+\tfrac{1}{3}\pi\sum(p-2)(\pi-\alpha)-\tfrac{2}{3}\sum(\pi^2-\psi)$$

for the content of any convex polytope in 4-dimensional spherical space. The first summation is over the 2-dimensional faces; such a face is a p-gon, and α is the dihedral angle between the two bounding hyperplanes that meet there. The second summation is over the vertices, ψ being the hypersolid angle at a vertex. [Compare L. Schläfli, *Gesammelte mathematische Abhandlungen, Band II*, Verlag Birkhäuser, Basel, 1953, pp. 164–190; MR **14**, 833.] *H. S. M. Coxeter* (Portland, Ore)

11313:

Busemann, Herbert. Volumes and areas of cross-sections. Amer. Math. Monthly **67** (1960), 248–250.

B_1, B_2 are convex bodies in 3-space which have a common centre z. $A(B_1, w)$ and $A(B_2, w)$ denote the areas of the regions in the plane through z, with unit normal vector w, within B_1, B_2, respectively. It is not known if $A(B_1, w)>A(B_2, w)$ for all w implies that the volume $V(B_1)$ of B_1 is greater than the volume $V(B_2)$ of B_2. Two examples are given in each of which the above conditions on B_1, B_2 are weakened. In both of these $A(B_1, w)>A(B_2, w)$ for all w but $V(B_1)<V(B_2)$. In the first example B_1, B_2 are star-shaped bodies with respect to the common centre z. In the second example B_1, B_2 are convex but the common point z is not a centre of either body.

D. Derry (Vancouver, B.C.)

11314:

Valentine, F. A. Characterizations of convex sets by local support properties. Proc. Amer. Math. Soc. **11** (1960), 112–116.

For a subset S of a linear Hausdorff space L, the author establishes several characterizations of convexity in terms of local properties. A boundary point x of S is called a point of "mild convexity" of S provided x is not the midpoint of any nondegenerate segment which lies, except for x, interior to S. The neatly proved Theorem 3 asserts that if S is an open connected set in L and if each boundary point of S is a point of mild convexity, then S is convex. This extends a result of Tietze [J. Reine Angew. Math. **160** (1929), 67–69] and clarifies the relationship between Tietze's result and an earlier theorem of Leja and Wilkosz [Ann. Soc. Polon. Math. **2** (1924), 222–224]. Theorem 3 leads to a similar characterization of convexity for closed connected sets (which employs the notion of a line "piercing" a set), and also to an extension of Tietze's characterization of convexity in terms of local support properties. *Victor Klee* (Seattle, Wash.)

11315:

Motzkin, Theodore S. Convex type varieties. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 1090–1092.

For varieties V of dimension m in affine space of dimension $n>1$, the following properties are defined: M_e (exact minimal order): V intersects every $(n-m)$-flat in at most $n-m+1$ points. U (unilaterality): for any point x on V, $V-\{x\}$ lies in an open halfspace bounded by a hyperplane H_x which is the flat of highest contact at x. M (minimal order): V intersects almost every $(n-m)$-flat in at most $n-m+1$ points. U_1 (local unilaterality): for

every x on V there exists a neighborhood N of x such that $V \cap N - \{x\}$ lies in an open halfspace bounded by H_x. F (flexion): for any x on V, $V \cap H_x = \{x\}$. F_1 (local flexion): for any x on V there exists an N such that $V \cap N \cap H_x = \{x\}$. Further B and S denote boundness and simplicity (non-self-intersection) in case of varieties without a lower-dimensional boundary. The author proves: (a) for every variety with property M_e, it is the case that $m=1$ or $m=n-1$; (b) there are varieties with property U_1 if and only if $n=\binom{m+k}{m}$, where k is odd; moreover for these n and m there exist varieties with properties U, B and S; (c) there exist varieties with property F_1 and without property U_1 if and only if $m=1$ and n is odd. None of these varieties has property B; all have properties F and S. Some examples are given.

L. A. Santaló (Buenos Aires)

11316:

Barthel, Woldemar. Zur isodiametrischen und isoperimetrischen Ungleichung in der Relativgeometrie. Comment. Math. Helv. **33** (1959), 241–257.

Dans R^n soit E un corps convexe borné, contenant l'origine O comme point intérieur, que l'on prend comme sphère étalon; on désigne par E^* le symétrique de E par rapport à O; tout corps déduit de E [ou de E^*] par une translation et une homothétie de rapport positif est appelé une sphère [ou une sphère réfléchie] de centre l'image de O par la translation utilisée. On appelle distance d'un point p_0 à un point p_1 la quantité: $\rho(p_0, p_1) = \inf_r (p_1 \in p_0 + rE)$; la distance maximum de deux ensembles K_0 et K_1 est la quantité:

$$d(K_0, K_1) = \sup_{p_0 \in K_0,\, p_1 \in K_1} \rho(p_0, p_1);$$

le diamètre $d(K)$ d'un ensemble K est la quantité $d(K, K)$. On désigne par $|K|_n$ le volume d'un ensemble mesurable K. Lorsque K_0 et K_1 sont convexes et que $|K_0|_n > 0$ on a $|K_0|_n^{1/n} + |K_1|_n^{1/n} \leq d(K_0, K_1)|E|_n^{1/n}$, l'égalité n'ayant lieu que si K_0 est une sphère réfléchie et K_1 une sphère de même centre. De là l'inégalité isodiamétrique, $2|K|_n^{1/n} \leq d(K)|E|_n^{1/n}$, valable pour tout ensemble mesurable K, l'égalité n'ayant lieu, lorsque K comprend plus d'un point, que si $E = E^*$ et si K est une sphère.

L'aire extérieure et l'aire intérieure de Minkowski étant définies par

$$S_+(K) = \liminf_{r \to +0} \frac{|K + rE|_n - |K|_n}{r},$$

$$S_-(K) = \liminf_{r \to +0} \frac{|K|_n - |K - rE^*|_n}{r},$$

on a les inégalités isopérimétriques:

$$S_+(K),\, S_-(K) \geq n|K|_n^{(n-1)/n}|E|_n^{1/n}$$

et il s'agit d'élucider le cas de l'égalité. Pour une direction P de plan à $(n-1)$ dimensions, soit x une direction de droite non parallèle à P, les abscisses sur cette direction étant aussi notées x; on désigne par $B(x_1, x_2)$ la bande des plans de direction P tels que $x_1 < x < x_2$; on pose:

$$\alpha_1(P) = \sup_x \{|K \cap B(-\infty, x)|_n = 0\},$$

$$\alpha_2(P) = \inf_x \{|K \cap B(x, +\infty)|_n = 0\};$$

et on appelle partie importante K_I de K l'intersection des bandes $B[\alpha_1(P), \alpha_2(P)]$ lorsque P varie. Alors, pour que l'égalité ait lieu dans la relation ci-dessus pour $S_+(K)$, il faut que K_I soit une sphère et que $S_+(K) = S_+(K_I)$; pour qu'elle ait lieu pour $S_-(K)$ il faut que K_I soit une sphère réfléchie.

L'auteur reprend une partie de la démonstration de Busemann pour $S_+(K)$; il en tire des remarques ingénieuses et utiles pour établir le résultat nouveau pour $S_-(K)$.

J. Favard (Paris)

DIFFERENTIAL GEOMETRY

See also 10990, 11099.

11317:

Фавар, Ж. [Favard, J.]. ★Курс локальной дифференциальной геометрии [Cours de géométrie différentielle locale]. Translated from the French by Yu. A. Rožanskaya and S. P. Finikov; preface by S. P. Finikov. Izdat. Inostr. Lit., Moscow, 1960. 559 pp. 25.25 r.

A translation into Russian of Favard's *Cours de géométrie différentielle locale* [Gauthier-Villars, Paris, 1957; MR **18**, 668].

11318:

Gołab, S. Le trièdre de Frenet aux points d'inflexion d'une courbe. Ann. Polon. Math. **9** (1960/61), 201–209.

The author shows that a regular curve of class C_2 in euclidean 3-space need not have a continuous moving trihedron. He gives a sufficient condition for the continuity of the latter at a given point.

P. Scherk (Toronto)

11319:

Saban, Giacomo. Sopra alcune curve notevoli. Boll. Un. Mat. Ital. (3) **15** (1960), 34–37. (English summary)

The author gives two extensions of the result, due to G. Pozzolo Ferraris [same Boll. **12** (1957), 46–49; MR **19**, 574], according to which the tangent to the locus described by any point P^* fixed in the osculating plane at a point P of a given skew curve $\mathscr{C}$, when P describes $\mathscr{C}$, stays perpendicular to the straight line obtained by joining P^* to the center of curvature of $\mathscr{C}$ in P.

M. Piazzolla-Beloch (Ferrara)

11320:

Backes, F. Sur la conservation des courbes $\mathscr{A}$ sur les surfaces orthogonales à un cercle cyclique. Acad. Roy. Belg. Bull. Cl. Sci. (5) **46** (1960), 604–608.

En prolongement d'un travail antérieur [même Bull. **41** (1955), 723–733; MR **17**, 403] relatif aux courbes (dites $\mathscr{A}$) tracées sur une surface, et définies par la propriété d'être tangentes aux caractéristiques d'une famille de sphères tangentes à la surface, l'auteur envisage le cas où, moyennant un choix convenable des sphères en question, les courbes $\mathscr{A}$ se correspondent sur les surfaces orthogonales au cercle générateur d'un système cyclique. Par l'emploi de la méthode du pentasphère orthogonal mobile de Demoulin, il est ainsi conduit à la proposition suivante: Si, pour chacune des familles de sphères focales des cercles du système cyclique considéré, les courbes $\mathscr{A}$

se correspondent sur les deux nappes de la famille de sphères envisagée, le cercle générateur C du système reste orthogonal à une sphère fixe Σ. En outre, les courbes $\mathscr{A}$ se correspondent sur toutes les surfaces orthogonales aux ∞^2 positions du cercle, les courbes étant celles relatives aux sphères tangentes à ces surfaces et coupant orthogonalement Σ. La proposition énoncée a lieu, en particulier, lorsque le cercle C engendre une congruence W, c'est-à-dire, lorsque ses coordonnées satisfont à une même équation aux dérivées partielles linéaire du 2ème ordre.

P. Vincensini (Marseille)

11321:

Mineo, Massimo. Superficie cón sistema geografico ortogonale isotermo o isotermo-coniugato. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **26** (1959), 368–371.

Geographic coordinates on a surface (with respect to a given line and a plane through it) have been introduced by C. Mineo [Giorn. Mat. Battaglini **49** (1910), 185–229]. The aim of this paper is to find all surfaces admitting a geographic orthogonal system. Some cases were known; all the others are determined. *E. Bompiani* (Rome)

11322:

Bogdănescu, V. On some properties of ruled surfaces. Lucrările Inst. Petrol Gaze Geol. București **5** (1959), 373–381. (Romanian. Russian and English summaries)

Let (S) be a non-developable ruled surface whose generators (D) are parallel to a fixed plane. Two properties are obtained: (1) The locus of tangents to the asymptotic lines of (S) in the points of (D) is a hyperbolic paraboloid. (2) If two asymptotic lines of (S) have constant torsion, all asymptotic lines have constant torsion and (S) is a right helicoid. *R. Blum* (Kingston, Ont.)

11323:

Vaisman, Izu. Quelques observations concernant les surfaces et les variétés non holonomes de l'S_3 euclidien. Acad. R. P. Romîne. Fil. Iași. Stud. Cerc. Ști. Mat. **10** (1959), 121–128. (Romanian. Russian and French summaries)

Between the normal curvature $1/R$ and the geodesic torsion $1/T$ corresponding to a direction θ in a regular point P on a surface (S) in euclidean 3-space there is the relation

$$\frac{1}{R^2}+\frac{1}{T^2}-H\frac{1}{R}+K=0,$$

where H and K are the mean and total curvature in P respectively. This relation can be interpreted as a circle (C) in the $(1/R,\,1/T)$-plane. One can, therefore, establish a one-to-one correspondence between the directions in P and the points on C, provided P is not umbilical on S. Based on this fact, noticed by R. Miron, the author makes some observations and derives a number of formulae connecting $1/R$ and $1/T$. He then generalizes these results to a non-holonomic $V_3{}^2$, in which case the circle (C) gets also a term in $1/T$. *R. Blum* (Kingston, Ont.)

11324:

Vincensini, Paul. Sur une transformation géométrique permettant de transformer, les unes dans les autres, les solutions des équations aux dérivées partielles $r+t=0$ et $rt-s^2=1$. Rend. Sem. Fac. Sci. Univ. Cagliari **28** (1958), 130–141.

A transformation $T(0,\alpha)$ for a rectilinear congruence is described and then the congruences of type (N), normal congruences which remain normal under transformations $T(0,\alpha)$. If a congruence of type (N) is referred to a spherical representation where the parameters for the sphere have been so chosen that the first fundamental form is $ds^2=E(u,v)(du^2+dv^2)$, then the congruence can be described by a function $\Phi(u,v)$ which satisfies

$$(1)\qquad \frac{\partial^2\Phi}{\partial u^2}+\frac{\partial^2\Phi}{\partial v^2}=0.$$

On the other hand, when a rectilinear congruence (N) is subjected to an inversion about the pole O one gets a special cyclic system (C) of circles. Such a special cyclic system of circles can be described by a function $\Omega(\varphi,\psi)$ which satisfies

$$(2)\qquad \frac{\partial^2\Omega}{\partial\varphi^2}\frac{\partial^2\Omega}{\partial\psi^2}-\left(\frac{\partial^2\Omega}{\partial\varphi\partial\psi}\right)^2=1.$$

This makes it possible to write out analytic instructions for going from a solution of (1) to a solution of (2), and indeed to get the most general solution this way. The author illustrates his work with the special case where the function E above is taken to be $1/\cosh^2 u$.

A. Schwartz (New York)

11325:

Godeaux, Lucien. Sur deux congruences W particulières. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 993–1005.

The congruences (of lines in 3-space) have a common focal surface. This is a Ribaucourt surface on which the asymptotic curves of both systems belong to linear complexes. On the second sheet of the focal surface of each congruence likewise the asymptotic curves belong to linear complexes.

A given congruence (j) has focal surface (x) on which the asymptotic lines of the two systems are u and v. Lines of the 3-space are represented by points of the Klein quadric Q in 5-space, j by J, and the tangents to u and v by U and V. The locus of J is a surface (J); on it, as a point traces a curve u [or v] on (x), the point J traces a curve u' [v']. The specialization of (j) is that the tangents to the curves u' [v'] at points of each curve v' [u'] are concurrent.

The three points J, U, V arising from the same point of (x) are collinear; the second congruence is represented by the points I which are the harmonic conjugates of J with respect to U, V. *T. G. Room* (Sydney)

11326:

Özkan, A. Integralfreie Darstellung der Zentralbewegung. Arch. Math. **11** (1960), 378–382.

In a projective space P_n consider a "moving" m-dimensional space E_m, describing a system ∞^1, such that the tangents to the trajectories at the points of an E_m pass through a point. This is called a "Zentralbewegung" by the author.

Obviously, two infinitely near E_m belong to an E_{m+1}

and intersect in an E_{m-1}. It follows that either the $\infty^1 E_m$ are the osculating spaces of a curve (edge of regression) or that they have a common subspace (in which case the problem is reduced to the preceding one in a smaller number of dimensions). It is apparent that these systems can be determined with no integration, using the edge of regression. *E. Bompiani* (Rome)

11327:

Švec, Alois. Généralisation d'une construction de M. B. Segre. Czechoslovak Math. J. **10** (85) (1960), 304–308. (Russian summary)

Study of correspondences between surfaces admitting a conjugate net, whose Laplace transforms degenerate, in a projective space S_{2n}. *E. Bompiani* (Rome)

11328:

Bargero-Rivelli, Elsa. Invarianti proiettivi di una calotta del 2° ordine e di un elemento curvilineo del 3° ordine. Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 253–275.

Determination of all the projective invariants belonging to the configuration of a surface cap of the second order and of a curvilinear differential element of the third order. *E. Bompiani* (Rome)

11329a:

Čakmazyan, A. V. Dual normalization. Akad. Nauk Armyan. SSR. Dokl. **28** (1959), 151–157. (Russian. Armenian summary)

11329b:

Čakmazyan, A. V. On dually normalized surfaces in Euclidean space. Akad. Nauk Armyan. SSR. Dokl. **29** (1959), 3–8. (Russian. Armenian summary)

A strip in P^n consists of hyperplane elements $x(u^1, \cdots, u^m)$, $\xi(u^1, \cdots, u^m)$, where the hyperplane ξ is tangent at $x(u)$ to the manifold M defined by $x(u)$ $(m<n-1)$. The strip is regular if $\det(h_{ij}) \neq 0$, where

$$h_{ij} = -\sum_k \frac{\partial x^k}{\partial u^i}\frac{\partial \xi^k}{\partial u^j}.$$

Let T_m be the tangent plane of M and τ_{n-m-1} the characteristic of the osculating plane of M. Following the procedure of A. P. Norden in *Prostranstva affinnoĭ svyaznosti* [Gosudarstv. Izdat. Tehn-Teor. Lit., Moscow, 1950; MR **12**, 441; the book deals principally with the case $m=n-1$] a regular strip can be dually normalized such that (i) its normal of the first kind is a P^{n-m}, passing through x and τ_{n-m-1} but containing no other point of T_m than x, and (ii) the normal of the second kind is a $P^{m-1} \subset T_m - x$. The first paper develops the analytical apparatus for such strips.

The second paper applies this apparatus to the following situation: $m=n-2$ and $M \subset E^n$; moreover, M is the basic manifold of a dually normalized regular strip such that the dual normalization coincides with the natural euclidean, which means in particular that $P^{n-m}=P^2$ is the normal plane to M. Let y be a unit vector in the direction of τ_{n-m-1}, and put

$$k_{ij} = -\sum_k \frac{\partial x^k}{\partial u^i}\frac{\partial y^k}{\partial u^j}.$$

Then M is determined up to motion by the g_{ij} (induced by E^n), h_{ij} and k_{ij}. However, h_{ij} and k_{ij} are not independent; the relations connecting them are given. A similar theorem holds for the spherical image of M defined by the unit normal to the osculating hyperplane of M. That these M are of a very special nature can be seen from the case $n=4$, $m=2$, where the author shows that M depends on two arbitrary functions of one and one arbitrary function of two variables.
H. Busemann (Los Angeles, Calif.)

11330:

Valette, Guy. La notion d'élément de contact pour les rubans de surface. Bull. Soc. Math. Belg. **11** (1959), 74–99.

Let σ be a curve in a differentiable manifold V. A strip along σ is a field of two-dimensional tangent planes along σ that contains the tangent vector field to σ. The author discusses various notions of order of contact of two strips, following Ehresmann's theory of jets, and announces many results concerning the action of the group of inversible jets on the space of elements of such strips in the case dim $V=3$.
R. Hermann (Belmont, Mass.)

11331:

Haimovici, Mendel. Sulla decomposizione dei sistemi differenziali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **23** (1957), 379–386.

Soit S un système différentiel extérieur, fermé, de genre p, au sens de Cartan-Kaehler. Soit S_1 un système partiel de S, fermé et de genre r. La restriction de S, à une variété intégrale régulière V_r de S_1, est fermée et désignée par S_2'. L'auteur donne un procédé pour déterminer le genre ρ et les caractères de S_2'. Par définition, S_1 fournit une décomposition régulière de S si toute variété intégrale V_β, $\beta \leq \rho$, de S est contenue dans une variété régulière V_r de S_1 et si, de plus, V_β est régulière pour la restriction S_2' de S sur V_r. Elle est dite complète si $\beta=\rho=p$. Pour une telle décomposition la recherche des intégrales régulières, de dimension β, du système S, se ramène à l'intégration successive du système S_1 et du système S_2'. L'auteur fournit quelques critères assurant l'existence d'une décomposition de ce type.
Th. Lepage (Zbl **80**, 76)

11332:

Gu, Čao-Hao. On pairs of connections and integral manifolds of systems of second order partial differential equations. I, II, III. Acta Math. Sinica **6** (1956), 153–162, 163–169, 426–432. (Chinese. Russian summary)

The purpose of these papers is to study the geometry of the integral manifolds of the system of second-order partial differential equations

$$(*)\qquad \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + H_{\alpha\beta}{}^i(u^\lambda, x^j, p_\gamma{}^l) = 0,$$

where $i, j, l = 1, \cdots, n$; $\alpha, \beta, \gamma, \lambda = 1, \cdots, m$; $p_\alpha{}^i = \partial x^i/\partial u^\alpha$, $H_{\alpha\beta}{}^i = H_{\beta\alpha}{}^i$, $n>m$. The main result in parts I and II is as follows. A suitable pair of connections can be chosen such that in a fixed sense the tangent plane elements of the integral manifolds of the system (*) are self-parallel, the

second connection can be considered as the derived connection of the first one, and the choice of the second connection is arbitrary. In part III necessary and sufficient conditions for the system (*) to be projectively flat are obtained, provided that the system (*) is completely integrable, the integral manifolds of the system (*) are independent of the choice of the base vectors of the initial plane elements, and $H_{\alpha\beta}{}^i$ have derivatives of higher orders.
C.-C. Hsiung (Bethlehem, Pa.)

11333:

Wright, J. Edmund. ★Invariants of quadratic differential forms. Reprinting of Cambridge Tracts in Mathematics and Mathematical Physics, No. 9. Hafner Publishing Co., New York, 1960. vii+90 pp. $3.00.

A reprinting of the 1908 work [Cambridge Univ. Press, London].

11334:

Thomas, Tracy Y. ★Concepts from tensor analysis and differential geometry. Mathematics in Science and Engineering, Vol. 1. Academic Press, New York-London, 1961. vii+119 pp. $5.00.

This book is intended to give an introductory account of tensor calculus and differential geometry, suitable for a one-semester course at the graduate level, for students of pure mathematics as well as for those students whose primary interest is the study of certain aspects of applied mathematics including the theory of relativity, fluid mechanics, elasticity, and plasticity theory.

From the point of view of a pure mathematician, parts of the text are rather loosely written. For example, on page 2 we read that if $y^i = f^i(x^1, x^2, \cdots, x^n)$, $x^i = \phi^i(y^1, y^2, \cdots, y^n)$ $(i=1, 2, \cdots, n)$ are analytical expressions of a one-one correspondence between coordinates x^i and y^i, then it follows from a well-known result in analysis that if the functions f^i are of class C^u, then so are the functions ϕ^i. The example $f(x)=x^3$, $\phi(y)=y^{1/3}$ shows that this "well-known" result is false. On the same page there is a "proof" that the functional determinants $|\partial y^i/\partial x^j|$ and $|\partial x^i/\partial y^j|$ are everywhere non-zero. The same example shows that this is also false.

The outlook of the book is far from modern. For example, vectors and tensors are defined as entities whose components transform in a specified way when coordinate-systems are changed. No attempt is made to distinguish tensor algebra from tensor calculus.

There is a useful section on normal coordinates, a topic not often dealt with in elementary texts. This leads to covariant differentiation and normal tensors.

The second part of the book deals with applications of tensor calculus to the (local) classical differential geometry of 2-dimensional surfaces imbedded in a 3-dimensional riemannian space, the latter sometimes being specialised to be euclidean. No applications are made to the theory of relativity, fluid mechanics, elasticity, or to plasticity theory.
T. J. Willmore (Liverpool)

11335:

Jankiewicz, C. Sur une généralisation du théorème de Noether. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 7 (1959), 409–411. (Russian summary, unbound insert)

Suppose that there is given a continuous group of order ρ of transformations in an n-dimensional space. The theorem of Noether referred to in the title is: If a geometric comitant with one component is a scalar density of weight $+1$ and if it satisfies equations of Euler-Lagrange, then there exist ρ laws of conservation expressed in the form of divergence. The author tries to generalize this theorem and obtains: Each geometric comitant with M components satisfying equations of Euler-Lagrange admits $\rho \cdot M$ generalized laws of conservation.
K. Yano (Seattle, Wash.)

11336:

Stong, Robert E. Some characterizations of Riemann n-spheres. Proc. Amer. Math. Soc. 11 (1960), 945–951.

Following the reviewer's joint paper with Feeman [Amer. J. Math. 81 (1959), 691–708; MR 21 #5990], the author gives some new characterizations of Riemann n-spheres. A typical result can be stated as follows. Let V^{n+1} be a Riemannian manifold of dimension $n+1 \geqq 3$ and constant Riemannian curvature such that there is a normal coordinate system of Riemann at a fixed point covering the whole manifold, V^n a closed orientable hypersurface of class C^3 imbedded in V^{n+1}, and M_i $(1 \leqq i \leqq n)$ the ith mean curvature of V^n at a general point. If there are integers s and i, $1 \leqq i < s \leqq n$, and constants $c_j \geqq 0$ for $i \leqq j \leqq s-1$ such that $M_i, \cdots, M_s > 0$ and $M_s = \sum_{j=i}^{s-1} c_j M_j$ at all points of V^n, then V^n is a Riemann n-sphere. Similar results are also obtained for the case in which $n+1 \geqq 4$ and V^{n+1} is not of constant Riemannian curvature.
C.-C. Hsiung (Bethlehem, Pa.)

11337:

Nakai, Mitsuru. Algebras of some differentiable functions on Riemannian manifolds. Japan. J. Math. 29 (1959), 60–67.

Let R be a Riemannian manifold of class C^m $(1 \leqq m \leqq \infty)$ with a metric tensor g_{ij} $(i, j=1, 2, \cdots, n)$ of class C^{m-1}, where n is the dimension of R. Neither connectedness nor the second countability axiom is assumed. Denote by $A^m(R)$ the set of all real-valued functions f on R of class C^m and having finite norms $\|f\|$, where the norm is

$$\|f\| = \sup_R |f| + \sup_R \left(g_{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \right)^{1/2}.$$

Repeated indices denote summation. An isometry from R to a manifold $'R$ is defined to be a differentiable homeomorphism T from R to $'R$ of class C^m, where T^{-1} is of the same class, such that $g^{ij}(\partial y^k/\partial x^i)(\partial y^l/\partial x^j) = {'g}^{kl}$, where $y^k = y^k(x^1, x^2, \cdots, x^n)$ $(k=1, 2, \cdots, n)$ is the local representation of T in terms of local coordinates and g_{ij} and $'g_{ij}$ are metric tensors of R and $'R$ respectively. If T is an isometry from R to $'R$, then T^{-1} is an isometry from $'R$ to R. An isometric isomorphism σ from $A^m(R)$ onto $A^m('R)$ is a norm-preserving isomorphism from $A^m(R)$ onto $A^m('R)$, the latter being considered as algebras over the real number field. The author's main theorem is: An isometry T from R onto $'R$ induces an isometric isomorphism $f \to f \circ T^{-1}$ of $A^m(R)$ onto $A^m('R)$; and, conversely, for each isometric isomorphism σ from $A^m(R)$ onto $A^m('R)$ there exists an isometry T which induces σ, i.e., $f^\sigma = f \circ T^{-1}$.

This theorem may be considered to be an extension of a result obtained by S. B. Myers [Proc. Amer. Math.

Soc. **5** (1954), 917–922; MR **16**, 491] who proved that the structure of any compact Riemannian manifold R of class C^m $(1 \leqq m < \infty)$ is completely determined by the structure of $A^m(R)$ as a normed algebra.

L. E. Pursell (Grinnell, Iowa)

11338:

Ishihara, Shigeru; Tashiro, Yoshihiro. On Riemannian manifolds admitting a concircular transformation. Math. J. Okayama Univ. **9** (1959/60), 19–47.

A conformal transformation of a Riemannian manifold M with metric tensor g_{ji} into a Riemannian manifold $'M$ with metric tensor $'g_{ji}$ is said to be concircular if it carries any geodesic circle of M into a geodesic circle of $'M$. A conformal transformation $'g_{ji} = \rho^2 g_{ji}$ is concircular if and only if

$$\rho_{ji} \equiv \nabla_j \rho_i - \rho_j \rho_i + \tfrac{1}{2} g_{ji} \rho_a \rho^a = \phi g_{ji} \qquad (\rho_i = \nabla_i \rho)$$

[K. Yano, Proc. Imp. Acad. Tokyo **16** (1940), 195–200, 354–360, 442–448, 505–511; **18** (1942), 446–451; MR **2**, 165, 303; **7**, 330].

A point of M is called a stationary point or an ordinary one of a concircular transformation (c.t.) according as ρ_i vanishes at the point or not. In a neighborhood of an ordinary point, the integral curve of ρ^h is called a ρ-curve and the hypersurface $\rho = \text{const}$ is called a ρ-hypersurface.

The authors first prove (Theorem 1): If a Riemannian manifold M admits a c.t. into a Riemannian manifold $'M$, then, for any ordinary point of transformation, there exists a coordinate neighborhood $\mathscr{U}$ of the point, where we can choose a system of coordinates u^h having the following properties. The function ρ depends only on the nth variable u^n in $\mathscr{U}$. The line-element of M is given by $ds^2 = \tau'^2 f_{cb}(u^a) du^c du^b + (du^n)^2$ where $\tau = 1/t$. The hypersurfaces defined by $u^n = \text{const}$ are ρ-hypersurfaces and the curves defined by $u^a = \text{const}$ are ρ-curves, and u^n indicates the arc length of ρ-curves.

The authors then study a compact Riemannian manifold admitting a c.t. and prove the following lemma: If a compact Riemannian manifold M admits a c.t., then M is differentiably homeomorphic to an n-dimensional sphere S_n and there exist exactly two stationary points O and O' in M. When S_n is represented by the unit hypersurface $(x^1)^2 + \cdots + (x^{n+1})^2 = 1$ in an $(n+1)$-dimensional Euclidean space E_{n+1}, the homeomorphism θ of M onto S_n maps a ρ-hypersurface on a sphere $(x^1)^2 + \cdots + (x^{n+1})^2 = 1$, $x^{n+1} = c$, $-1 < c < -1$, and a ρ-curve on a great circle passing through the antipodal points $(0, \cdots, 0, 1)$ and $(0, \cdots, 0, -1)$, which are images of the stationary points. From this lemma, the authors derive Theorem 2: If a compact Riemannian manifold M admits a c.t., then it is conformally homeomorphic to an n-dimensional spherical space of curvature 1. The homeomorphism of M onto the unit sphere S_n in E_{n+1} is given by the mapping θ in the above lemma. The ratio of the metric tensor at a point P of M to that at the corresponding point of S_n by θ is constant when P moves in a ρ-hypersurface of M. Conversely, if a compact Riemannian manifold M is conformally homeomorphic to S_n in this way, then M admits a c.t.

The authors study next a complete Riemannian manifold of constant scalar curvature admitting a c.t., and prove Theorem 3: We assume that a complete Riemannian manifold M of constant scalar curvature k admits a c.t. into a Riemannian manifold $'M$ of constant scalar curvature $'k$. Then M is isometrically homeomorphic (I) to an n-dimensional Euclidean space if $k = 0$, (II) to an n-dimensional spherical space if $k > 0$, (III) to an n-dimensional hyperbolic space if $k < 0$. Theorem 4: Let M and $'M$ be complete Riemannian manifolds of constant scalar curvature. If there exists a non-homothetic c.t. of M onto $'M$, then the scalar curvatures are positive and both M and $'M$ are isometrically homeomorphic to spherical spaces, and conversely.

In the last section, the authors study the local homogeneous holonomy group of a Riemannian manifold admitting a c.t. and prove several theorems similar to Theorem 5: If a complete, non-flat Riemannian manifold admits a c.t., then its local homogeneous holonomy group at any point is SO(n).

K. Yano (Seattle, Wash.)

11339:

Tashiro, Yoshihiro. Remarks on a theorem concerning conformal transformations. Proc. Japan Acad. **35** (1959), 421–422.

The author proves first Theorem 1: In order that a conformal transformation map an Einstein manifold into an Einstein manifold, it is necessary and sufficient that the transformation be concircular. The sufficiency has been already proved by the reviewer [Proc. Imp. Acad. Tokyo **18** (1942), 446–451; MR **7**, 330]. The author combines this theorem with previous results [S. Ishihara and Y. Tashiro, #11338], and proves Theorem 2: If a complete Einstein manifold M is transformed conformally into an Einstein manifold $'M$, then the manifold M is (1) a Euclidean space if $k = 0$, (2) a spherical space, if $k > 0$, or (3) a hyperbolic space if $k < 0$, where k is the curvature of the space. Theorem 3: If a complete Einstein manifold M admits a conformal transformation onto itself, then the manifold M is a spherical space. These results generalize a theorem of the reviewer and Nagano [Ann. of Math. (2) **69** (1959), 451–461; MR **21** #345].

K. Yano (Seattle, Wash.)

11340:

Knebelman, M. S.; Yano, K. On homothetic mappings of Riemann spaces. Proc. Amer. Math. Soc. **12** (1961), 300–303.

The authors prove the following two theorems. (Theorem 2) In a Riemannian space with non-zero constant curvature scalar, an (infinitesimal) homothetic transformation is an (infinitesimal) isometry. (Theorem 4) In a compact orientable Riemannian space with negative constant curvature scalar, an infinitesimal conformal transformation is an infinitesimal isometry. Theorem 1 is a special case of Theorem 2. Theorem 3 is equivalent to Theorem 2. In both Theorems 2 and 4, the authors use infinitesimal transformations. The proof of Theorem 2 works, with a trivial modification, for any homothetic transformation. It is not clear, however, whether Theorem 4 holds for conformal transformations not belonging to the connected component of the identity of the group of conformal transformations. For a complete Riemannian manifold, Theorem 2 is a special case of the reviewer's result in Nagoya Math. J. **9** (1955), 39–41 [MR **17**, 892].

S. Kobayashi (Vancouver, B.C.)

11341:

Yano, Kentaro; Nagano, Tadashi. The de Rham decomposition, isometries and affine transformations in Riemannian spaces. Japan. J. Math. **29** (1959), 173–184.

In this paper, the authors first define a de Rham decomposition of the tangent space $T(x)$ at a point x of a Riemannian manifold M in the following way. Denoting by H the homogeneous holonomy group of M at x and by H^0 the identity component of H, a direct sum decomposition $T(x)=\sum_{\alpha=0}^{r} T_\alpha$ is called a de Rham decomposition if the subspaces T_α are mutually orthogonal, every T_α is invariant by H, every T_α, $\alpha>0$, is irreducible and if T_0 is the maximal subspace on which H^0 is trivial. G. de Rham [Comment. Math. Helv. **26** (1952), 328–344; MR **14**, 584] originally defined such a decomposition when M is simply connected and showed that if M is furthermore complete, M is isometric to the direct product of maximal integral manifolds M_α of parallel distributions obtained from the subspaces T_α. In this case, the Lie algebra of all affine fields on M (vector fields which generate affine transformations) is naturally isomorphic with the direct product of the Lie algebras of affine fields on M_α [J. Hano, Nagoya Math. J. **9** (1955), 99–109; MR **17**, 891]. The same holds for the Lie algebra of Killing vector fields (vector fields which generate isometries).

The authors prove that the de Rham decomposition in their sense for an arbitrary Riemannian manifold M is unique up to order and then proceed to study the situations concerning affine fields or Killing vector fields on M with respect to this tangent space decomposition. A typical result can be stated as follows. From $T(x)=\sum_{\alpha=0}^{r} T_\alpha$, define a similar decomposition of $T(y)$ at every point y of M by parallel displacement and denote by P_α the field of projections $T(y)\to T_\alpha(y)$ at each point y. If v is an affine field on M, then $P_\alpha v$ is an affine field on M and its restriction to the integral manifold M_α of the parallel distribution T_α is an affine field of M_α. Various results, which are somewhat too technical to be reproduced here, can be obtained when M satisfies one or more of the assumptions such as analyticity, completeness, simple connectivity, and infinitesimal homogeneity.

K. Nomizu (Providence, R.I.)

11342:

Reinhart, Bruce L. Closed metric foliations. Michigan Math. J. **8** (1961), 7–9.

A closed metric foliation F is a foliation (or involutive distribution) on a Riemannian manifold M whose leaves (or maximal connected integral submanifolds) are closed, geodesically parallel subsets of M. This paper shows that M/F, the space of all leaves of F, is a V-manifold in the sense of Satake [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363; MR **18**, 144] and that the natural projection $M\to M/F$ is a V-fibre map. Roughly speaking, M/F is locally homeomorphic to the quotient of a Euclidean space by a finite differentiable transformation group.

The proof uses previous results of the author [Ann. of Math. (2) **69** (1959), 119–132; MR **21** #6004]. (He states that there is an error in his previous paper that can be corrected in the closed metric foliation case.) A noteworthy lemma states that the holonomy group of each leaf is finite and that the set of points of M lying on leaves whose holonomy group is the identity is open and dense in M. {Reviewer's remark: This result is also true for closed metric foliation with singularities, if the relevant concepts are correctly defined. For example, this is well-known in case the leaves are orbits of a compact differentiable transformation group. Indeed, much of the recent work on foliations is motivated by an attempt to extend theorems that are known in the transformation-group case.}

R. Hermann (Belmont, Mass.)

11343:

Teleman, C. Sur les variétés de Grassmann. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2 (50)** (1958), 202–224.

Soit $\Gamma=\Gamma_k^{p,q}$ $(p\leq q)$ la variété de Grassmann, c'est-à-dire, la variété des p-plans $\lambda_i^{\alpha}\xi^i=0$ $(i=1, 2, \cdots, n=p+q)$ de l'espace euclidien E_k^n à n dimensions et ayant les coordonnées dans le corps k, k étant le corps R des nombres réels, le corps K des nombres complexes ou le corps Q des quaternions. E. Cartan a étudié les espaces Γ, au point de vue riemannien global, et a montré que ce sont des espaces symétriques.

On se propose dans ce travail de développer les résultats de Cartan en insistant sur les propriétés locales des espaces riemanniens Γ et non pas sur leurs propriétés globales. Le travail se divise en deux chapitres.

Dans le premier chapitre, on obtient la métrique de Γ. Pour l'obtenir, on donne deux représentations de Γ; la première de ces représentations est donnée dans l'espace du groupe unitaire U de E_k^n; c'est une représentation analytique nonholonome. La seconde représentation applique Γ topologiquement et isométriquement dans l'espace euclidien unitaire $E_k^{n^2}$, mais l'image de Γ n'est une variété analytique que si l'on passe à l'espace euclidien réel E^{n^2}. C'est pourquoi, pour obtenir le transport parallèle dans Γ, on utilise la représentation nonholonome de Γ. On donne ensuite les équations des géodésiques de Γ.

Dans le second chapitre, on étudie les intégrales harmoniques des espaces riemanniens V_n, symétriques clos. Le cas des espaces V_n représentatifs des groupes simples classiques a été tranché par Hodge [*Theory and applications of harmonic integrals*, 2nd ed., Cambridge Univ. Press, London, 1952; MR **14**, 500]. On suppose que V_n n'est pas un tel espace, donc le groupe des mouvements de V_n est un groupe simple clos.

Soit G_r le groupe de V_n et g le sous-groupe de stabilité d'un point de V_n; on peut alors identifier V_n avec l'espace G_r/g des classes latérales à gauche xg, $x\in G_r$, de g dans G_r. Soit Γ^* l'espace nonholonome complémentaire orthogonal de l'espace holonome formé par les classes xg, l'orthogonalité étant considérée par rapport à la métrique de Weyl de G_r. On sait d'après Cartan que V_n est isométrique à Γ^*. Or, l'auteur montre que l'on peut obtenir tous les invariants intégraux de V_n en projetant orthogonalement sur Γ^* les intégrales de G_r, invariantes par rapport aux translations (*) $x\to\theta x$ $(\theta\in G_r)$, $x\to xc$ $(c\in g)$. Mais dans un espace symétrique, chaque invariant intégral est une intégrale harmonique, et inversement chaque intégrale harmonique est invariante par rapport aux mouvements de l'espace. On montre aussi que l'on peut prendre pour ω seulement les intégrales de différentielle exacte et on obtient encore l'ensemble de toutes les intégrales harmoniques de V_n.

Si l'espace de Riemann $V_n=G_r/g$ n'est pas symétrique, mais à groupe G_r clos, le théorème énoncé plus haut s'étend de la manière suivante: On peut obtenir toutes les formes invariantes de V_n en projetant orthogonalement sur Γ^* les formes ω de G_r, qui sont les invariants du groupe (*).

K. Yano (Seattle, Wash.)

11344:

Wakakuwa, Hidekiyo. On almost complex symplectic manifolds and affine connections with restricted homogeneous holonomy group Sp(n, C). Tôhoku Math. J. (2) **12** (1960), 175–202.

The author discusses affine connections on M_{4n} (a manifold of dimension $4n$) whose restricted homogeneous holonomy group is contained in the real representation of Sp$(n; C)$. The content would be better described by considering the following general setting. Let $\{T\}$ be a system of tensors over the real vector space R^m and G the group of linear transformations of R^m leaving $\{T\}$ invariant. The reduction of the structure group of the tangent bundle of M_m to G is equivalent to the existence of a system of tensor fields $\{t\}$ on M_m which is equivalent to $\{T\}$ at each point. A connection in such a reduced bundle is an affine connection which makes $\{t\}$ all parallel. The author considers the case $G=\mathrm{Sp}(n; C)$ and discusses the existence of affine connections without torsion in the reduced bundle by utilizing the Nijenhuis tensor. The results are too complicated to be stated here.

S. Kobayashi (Vancouver, B.C.)

11345:

Eum, Sang-Seup. Direct complexification of a Riemannian manifold into a Kaehlerian manifold. Kyungpook Math. J. **2** (1959), 53–60.

Consider a complex manifold covered by a system of complex coordinate neighborhoods $(z^a, z^{\bar a})$ and with a Riemannian metric $ds^2=g_{CB}dz^Cdz^B$. If we put $z^a=x^a+ix^{\bar a}$, $z^{\bar a}=x^a-ix^{\bar a}$, then ds^2 in this real coordinate system (x^h) takes the form $ds^2=b_{CB}dx^Cdx^B$. The author studies the conditions for b_{CB} in order that $ds^2=g_{CB}dz^Cdz^B$ be Hermitian or Kählerian. The author also discusses harmonic and Killing vectors in a compact Kählerian space.

K. Yano (Seattle, Wash.)

11346:

Goldberg, S. I. Conformal transformations of Kaehler manifolds. Bull. Amer. Math. Soc. **66** (1960), 54–58.

The author first proves two lemmas. (1) The harmonic forms on a compact orientable Riemannian manifold M are invariant under the Lie algebra of infinitesimal isometries on M. (2) The harmonic forms of degree $p=n/2$ of a compact orientable Riemannian manifold M of even dimension n are invariant under the Lie algebra of infinitesimal conformal transformations (i.c.t.) on M. From these lemmas, the author deduces two more lemmas. (3) If dim $M=2$, the inner product of a harmonic vector field and a vector field defining an i.c.t. is a constant on M. (4) An i.c.t. X of a 4-dimensional compact Kähler manifold M is necessarily a motion. Now the author's main theorem is as follows: The largest connected Lie group of c.t. of a $4k$-dimensional compact Kähler manifold coincides with the largest connected group of automorphisms.

K. Yano (Seattle, Wash.)

11347:

Tachibana, Shun-ichi. Note on conformally flat almost-Kählerian spaces. Nat. Sci. Rep. Ochanomizu Univ. **10** (1959), 41–43.

Consider an almost-Hermitian space, that is, an even-dimensional space with a Riemannian metric g_{ji} and an almost-complex structure $F_i{}^h$ satisfying

$$g_{cb}F_j{}^cF_i{}^b = g_{ji}, \quad F_{ji} = F_j{}^ag_{ai} = -F_{ij}.$$

If g_{ji} and $F_i{}^h$ satisfy

$$\nabla_jF_{ih}+\nabla_iF_{hj}+\nabla_hF_{ji} = 0,$$

∇_j denoting the covariant derivation with respect to Levi-Civita's parallelism, then the almost-Hermitian space is called an almost-Kaehlerian space. It is well known that the tensor F_{ji} in an almost-Kaehlerian space is harmonic. In this note, the author proves the following two theorems. (1) An almost-Kaehlerian space with the curvature tensor of the form

$$K_{kjih} = \alpha[(g_{kh}g_{ji}-g_{jh}g_{ki})+(F_{kh}F_{ji}-F_{jh}F_{ki})-2F_{kj}F_{ih}]$$

is Kaehlerian. (2) There does not exist a conformally flat almost-Kaehlerian space if the curvature scalar is positive.

K. Yano (Seattle, Wash.)

11348:

Tachibana, Shun-ichi. On infinitesimal holomorphically projective transformations in certain almost-Hermitian spaces. Nat. Sci. Rep. Ochanomizu Univ. **10** (1959), 45–51.

The space which the author considers here is an almost-Hermitian space in which the structure tensor F_{ji} is a Killing tensor. The author calls such a space a K-space. The infinitesimal transformation which the author studies here is a vector field v^h which satisfies

$$\underset{v}{\pounds}\left\{{h \atop j\;i}\right\} = \rho_j\delta_i^h+\rho_i\delta_j^h-\bar\rho_jF_i{}^h-\bar\rho_iF_j{}^h$$

where $\bar\rho_i=F_i{}^h\rho_h$. The author calls such a transformation a holomorphically projective transformation (for short, h.p.tr.). If the covariant derivative of the tangent of a curve is in the plane spanned by the tangent and the transform of it by the $F_i{}^h$, the curve is called a holomorphically planar curve. The author proves the following theorems. (1) In a K-space, in order that an infinitesimal transformation carries any planar curve into a planar curve, it is necessary and sufficient that it is an analytic h.p.tr. (2) If ρ_i is the associated vector of an analytic h.p.tr., then $\bar\rho^h$ is a Killing vector. (3) In an Einstein K-space with $R\neq0$, an analytic h.p.tr. v^h is decomposed into the form $v^h=p^h+F_i{}^hq^i$, where p^h and q^h are both Killing vectors and $F_i{}^hq_h$ is a gradient. The decomposition is unique. (4) In an Einstein K-space with $R\neq0$, the associated vector of an analytic h.p.tr. is a h.p.tr. (5) In a complete Einstein K-space with $R<0$, the length of the associated vector of an analytic h.p.tr. cannot be bounded. (6) In a compact Einstein K-space with $R\neq0$, the associated vector of an analytic h.p.tr. is also an analytic h.p.tr. (7) If an Einstein K-space with $R\neq0$ admits an analytic non-affine h.p.tr., then the restricted homogeneous holonomy group contains the full linear group $U(n/2)$. (8) In a K-space of positive constant curvature, an analytic h.p.tr. is necessarily affine.

K. Yano (Seattle, Wash.)

11349:

Tachibana, Shun-ichi; Ishihara, Shigeru. On infinitesimal holomorphically projective transformations in Kählerian manifolds. Tôhoku Math. J. (2) **12** (1960), 77–101.

In a Kählerian manifold M with complex structure F_i^h and Riemannian metric g_{ji}, a vector field is called an infinitesimal holomorphically projective transformation (i.h.p.t.) when it satisfies

$$\mathop{\pounds}_{v}\left\{ {h \atop j\ i} \right\} = \rho_j\delta_i^h + \rho_i\delta_j^h - \tilde{\rho}_j F_i^h - \tilde{\rho}_i F_j^h,$$

where $\tilde{\rho}_i = F_i^a\rho_a$. In an M, a curve whose curvature vector is contained in a plane spanned by its tangent and the transform of it by F_i^h is called a holomorphically planar curve (h.p.c.).

Devoting § 1 to preliminaries, the authors prove in § 2 that, in an M, an infinitesimal transformation preserves h.p.c. if and only if it is an i.h.p.t.

In § 3, they study some properties of i.h.p.t. and prove the following theorems. (1) In a compact M, an i.h.p.t. is analytic. (2) In an irreducible M admitting no quaternion structure, any i.h.p.t. is analytic. (3) In an irreducible M with non-vanishing Ricci tensor, any i.h.p.t. is analytic. (4) If a vector ρ_i is the associated vector of an analytic i.h.p.t., then ρ^h is analytic and $\tilde{\rho}^h$ is a Killing vector.

Now the holomorphically projective curvature tensor is defined as

$$P_{kji}^h = K_{kji}^h - (n+2)^{-1} \times (\delta_k^h K_{ji} - \delta_j^h K_{ki} + F_k^h S_{ji} - F_j^h S_{ki} + 2S_{kj}F_i^h),$$

where K_{kji}^h and K_{ji} are curvature tensor and Ricci tensor of M respectively and $S_{ji} = -\frac{1}{2}F^{ts}K_{tsji}$.

The authors study in § 4 an analytic i.h.p.t. which leaves invariant the covariant derivative of P, and prove that if an M admits an analytic non-affine i.h.p.t. which leaves invariant the covariant derivative of P, then M is a space of constant holomorphic curvature.

In § 5, they prove that if an M satisfying $\nabla_k K_{ji} = 0$ admits an analytic non-affine i.h.p.t., then it is a Kähler-Einstein manifold.

§ 6 is devoted to the study of Kähler-Einstein manifolds with non-vanishing scalar curvature K. They obtain the following theorems. (1) In a Kähler-Einstein manifold with $K \neq 0$, an i.h.p.t. is analytic if and only if its associated vector is analytic. (2) In a Kähler-Einstein manifold with $K \neq 0$, an analytic i.h.p.t. v^h is uniquely decomposed in the form $v^h = p^h + F_i^h q^i$, where p and q are both Killing vectors. (3) In a complete Kähler-Einstein manifold with $K > 0$, the length of the associated vector of an analytic non-affine i.h.p.t. is not bounded. (4) If a Kähler-Einstein manifold with $K \neq 0$ admits an analytic non-affine i.h.p.t. then its local homogeneous holonomy group at any point is the full unitary group $U(n/2)$. (5) If, in a Kähler-Einstein manifold with $K \neq 0$, the vector space consisting of all analytic gradient i.h.p.t. is transitive at each point, then the manifold is a space of constant holomorphic curvature.

In the last section, the authors prove that in a compact space of constant holomorphic curvature with $K > 0$, a necessary and sufficient condition for a vector field to be analytic is that it be an i.h.p.t. *K. Yano* (Seattle, Wash.)

11350:

Smith, J. Wolfgang. Fundamental groups on a Lorentz manifold. Amer. J. Math. **82** (1960), 873–890.

Let $\mathscr{L}$ denote a metric Lorentz structure on a differentiable manifold $\mathscr{V}$. For each base point $x \in \mathscr{V}$ and for each non-negative integer q the author defines the group $\tau_q(L, x)$, which depends only on x, q and the conformal Lorentz structure L. To form the group τ_q consider the collection of time-like loops based at x and having exactly q corners, at each of which both tangent vectors are time-like. A generalized q-loop is obtained from a q-loop by inserting or deleting "stings", where a sting is a path of the form ff^{-1}. The set of generalized q-loops generate T_q^*, which is normalized to T_q in a certain way, and the elements of τ_q are the homotopy classes of T_q, where the homotopy is performed within T_q. The obvious homomorphism into the fundamental group $i_q: \tau_q \to \pi_1$ and the homomorphism defined by the process of inserting stings $h_q: \tau_q \to \tau_{q+2}$ form a commutative diagram and enable one to define the direct limit groups $\tau_0 \to \tau_2 \to \tau_4 \to \cdots \to \tau_+$ and $\tau_1 \to \tau_3 \to \tau_5 \to \cdots \to \tau_-$. Moreover there exist corresponding homomorphisms $i_\pm: \tau_\pm \to \pi_1$. Theorem 4.1: i_+ $[i_-]$ maps τ_+ $[\tau_-]$ isomorphically onto the subgroup π_1^+ $[\pi_1^-]$, generated by loops of even [odd] parity. If L is time-orientable, $\pi_1^+ = \pi_1$ and π_1^-, τ_q (q odd) are trivial. Otherwise π_1^+ is a normal subgroup of index 2 and $\pi_1^- = \pi_1$.

The author defines the component $\{x\}$ of x to be those points $y \in \mathscr{V}$ for which there exist suitable isomorphisms of $\tau_q(L, y) \to \tau_q(L, x)$, $q \geqq 0$. Two theorems on the properties of $\{x\}$ are proved. The final result applies to flat Lorentz manifolds $\mathscr{V}$. Theorem 7.1: If $\mathscr{V}$ is flat and complete, then the maps h_0 are monomorphisms and the maps h_q, $q > 0$, are isomorphisms. *L. Markus* (Minneapolis, Minn.)

11351:

Chern, Shiing-Shen. Differential geometry and integral geometry. Proc. Internat. Congress Math. 1958, pp. 441–449. Cambridge Univ. Press, New York, 1960.

The author analyzes different examples in order to show how the methods of integral geometry may be useful to problems in differential geometry. (1) The measure of the spherical image of a closed submanifold in Euclidean space conduces to the theorems of Chern and Lashof [Amer. J. Math. **79** (1957), 302–318; Michigan Math. J. **5** (1958), 5–12; MR **18**, 927; **20** #4301] and to some related solved and unsolved questions which the author indicates. (2) Let $Z: M_n \to P_{n+N}$ be a complex analytic mapping of a complex manifold M_n of dimension n into the complex projective space P_{n+N}. If P_{n+N} is provided with an elliptic Hermitian metric, the volume $v(M_n)$ of $Z(M_n)$ may be computed. Many interesting questions arise from the comparison of $v(M_n)$ and the number of points of intersection of $Z(M_n)$ with one or more linear spaces L of P_{n+N} in general position [for the case $E_1 \to P_{1+N}$, E_1 = complex Euclidean line, see Ahlfors, Acta Soc. Sci. Fenn. A **3** (1941), no. 4, 1–31; MR **2**, 357]. (3) Integral formulae play an important role in the proofs of rigidity or uniqueness theorems. As an example, a new integral formula is indicated which reduces to a purely algebraic problem the proof of the following uniqueness theorem: If two closed strictly convex hypersurfaces are such that at points with parallel normals, the mth (for fixed $m \geqq 2$) elementary symmetric functions of the principal radii of curvature have the same value, then they differ from each other by a translation.

L. A. Santaló (Buenos Aires)

11352:

Klotz, Tilla. On G. Bol's proof of Carathéodory's conjecture. Comm. Pure Appl. Math. **12** (1959), 277–311.

The Carathéodory conjecture is that every closed, strictly convex, sufficiently smooth surface has at least two umbilic points. Since the mapping index is not 0, there is at least one umbilic point, and the Carathéodory conjecture follows from the assertion that the index of any umbilic point is not the total index. This deceptively simple assertion is apparently quite difficult to establish, and the known proofs require analytic surfaces. The Bol proof involves a deformation of a curve, but overlooks certain singular cases. The author completes Bol's argument by a careful analysis of these special cases.

D. G. Bourgin (Urbana, Ill.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 11082, 11096, 11099, 11100, 11387, 11390, 11391.

11353:

Polák, Václav. On a covering of all rational points in the plane by an infinite polygonal curve. Časopis Pěst. Mat. **85** (1960), 141–145. (Czech. Russian and English summaries)

The author considers a class of subsets of the plane introduced by A. F. Möbius [cf. *Gesammelte Werke*, Leipzig, 1885, 237–251] and constructs, for every set H of this class, a simple polygonal line $(\cdots A_{-1}A_0A_1\cdots)$ such that H is equal to the set of all A_{2i}.

M. Katětov (Prague)

11354:

Jerrard, R. P. Inscribed squares in plane curves. Trans. Amer. Math. Soc. **98** (1961), 234–241.

The author shows that, given a plane curve C defined by real analytic functions $x(t)$, $y(t)$ with period T, it is always possible to find four points on C that are the vertices of a square. In fact, in a certain sense, there is always an odd number of such squares. The argument depends upon defining a single-valued continuous function by selecting values of a many-valued function, and is partly topological. A similar result was proved by L. G. Šnirel'man [Uspehi Mat. Nauk **10** (1944), 34–44; MR **7**, 35].

H. G. Eggleston (London)

11355:

Lintz, Rubens G. Remarques sur deux travaux antérieurs. Ann. Mat. Pura Appl. (4) **52** (1960), 9–10.

It is remarked that the Theorem 2, in the author's paper, same Ann. (4) **43** (1957), 357–370 [MR **19**, 437] is false. As to its consequences, ibid. **46** (1958), 343–348 [MR **21** #323], these can be saved because usually "the accessible points of π are dense".

R. Arens (Los Angeles, Calif.)

11356:

Koch, R. J.; Krule, I. S. Weak cutpoint ordering on hereditarily unicoherent continua. Proc. Amer. Math. Soc. **11** (1960), 679–681.

The authors give two necessary and sufficient conditions for determining when an hereditarily unicoherent continuum admits a monotone, closed partial order with unique minimal element. This results in an improvement of the characterization of a class of partially ordered spaces, called generalized trees, as given by L. E. Ward, Jr. [same Proc. **8** (1957), 798–804; MR **20** #3516]. An example is given of an arcwise-connected hereditarily unicoherent continuum on which there does not exist a monotone closed partial order with unique minimal element.

Anne L. Hudson (New Orleans, La.)

11357:

Andrews, James J. A chainable continuum no two of whose nondegenerate subcontinua are homeomorphic. Proc. Amer. Math. Soc. **12** (1961), 333–334.

The author constructs the continuum as the inverse limit of chainable continua and presents a modification of the example given by Anderson and Choquet [same Proc. **10** (1959), 347–353; MR **21** #3819].

A. Lelek (Wrocław)

11358:

Lehner, G. R. Extending homeomorphisms on the pseudo-arc. Trans. Amer. Math. Soc. **98** (1961), 369–394.

Generalizing an earlier result of Bing [Duke Math. J. **15** (1948), 729–742; MR **10**, 261], the author proves that if A and B are closed subsets of the pseudo-arc M (an hereditarily indecomposable chainable continuum), if A consists of a finite number n of components and (*) M is irreducible between each pair of them as well as between each pair of components of B, then every homeomorphism h of A onto B can be extended to a homeomorphism of M onto M. In this theorem, the condition (*) can be omitted provided that the conclusion is restricted to such homeomorphisms h that for every subset X of A, consisting of exactly $m=\max\{2, n-1\}$ points, there exists a homeomorphism of M onto M, which agrees with h on X. The paper also contains a more detailed proof of a theorem on crooked chains, which has been an important point in Bing's paper mentioned above.

A. Lelek (Wrocław)

11359:

Debrunner, Hans; Fox, Ralph. A mildly wild imbedding of an n-frame. Duke Math. J. **27** (1960), 425–429.

By an n-frame D in Euclidean 3-space is meant a collection of n arcs $\alpha_i=pq_i$ in 3-space such that $\alpha_i\cap\alpha_j=p$, $i\neq j$. An n-frame is mildly wild if it is not tamely imbedded but each of the $(n-1)$-frames obtained by deleting all of one of the arcs α_i but p is tamely imbedded. This paper gives an example for each n, $n\geq 2$, of a mildly wild n-frame.

E. Dyer (Chicago, Ill.)

11360:

Kwun, Kyung Whan. A fundamental theorem on decompositions of the sphere into points and tame arcs. Proc. Amer. Math. Soc. **12** (1961), 47–50.

Bing [Ann. of Math. (2) **65** (1957), 363–374; MR **19**, 1187] has produced an upper semicontinuous decomposition of the 3-sphere S^3 into points and tame arcs whose quotient space is not a manifold. Rosen [Notices Amer. Math. Soc. **6** (1959), 641] has modified Bing's example so that the quotient space is nowhere Euclidean. The author proves that the quotient space is nevertheless a homotopy manifold. In general he proves the following theorem: Let G be an upper semicontinuous decomposition of S^n

with a finite-dimensional quotient space X such that each element g of G is an absolute retract having an open-cell complement in S^n. Then X is an n-dimensional homotopy manifold. *Morton Brown* (Princeton, N.J.)

11361:

Rosen, Ronald H. Decomposing 3-space into circles and points. Proc. Amer. Math. Soc. **11** (1960), 918–928.

Many interesting questions exist concerning the embedding of monotone decompositions of a Euclidean space in a Euclidean space. An effective general theory will apparently await the solution of a good many special problems of which this paper is typical. Bing and Curtis [same Proc. **11** (1960), 149–155; MR **22** #8468] have constructed an upper semicontinuous decomposition of Euclidean 3-space E^3 in which there are exactly twelve nondegenerate elements (all circles) such that the decomposition space is not embeddable in 4-space. Following a suggestion of the author, they reduce the number of nondegenerate elements to nine. In this paper the author further refines the methods and reduces the number of nondegenerate elements to six. His example consists of three large circles K_1, K_2 and K_3 in E^3 each of which links the other two and three small circles J_1, J_2 and J_3 such that J_i links K_i if and only if $i=j$ $(i, j=1, 2, 3)$. With these six circles as the only nondegenerate elements of the decomposition the author shows [by non-trivially extending and proving some lemmas due to Flores] that the resulting decomposition space contains "a singular image" of the Menger 2-dimensional polyhedron M_2 which cannot be embedded in E^4. The author conjectures that the number of circles can be reduced to five with the same result but not to four. The reviewer is of the opinion that the author can reduce the number of circles to four by replacing J_1, J_2 and J_3 by a single circle which links all three large circles. *F. B. Jones* (Chapel Hill, N.C.)

11362:

Dedecker, Paul. Introduction aux structures locales. Colloque Géom. Diff. Globale (Bruxelles, 1958), pp. 103–135. Centre Belge Rech. Math., Louvain, 1959.

An inductive category is one with a suitably defined notion of sub-object. On an object of such a category a paratopology is an appropriate collection of sub-objects. The objects-with-paratopology themselves form an inductive category. A contravariant class-valued functor on this last is a species of paralocal structure relative to the original category; a collatability condition makes it local. The Γ-structures of Haefliger [Comment. Math. Helv. **32** (1958), 248–329; MR **20** #6702] are adduced as an example. An appendix contains a program for re-edifying set theory. *A. Heller* (Urbana, Ill.)

11363:

Grimeisen, Gerhard. Gefilterte Summation von Filtern und iterierte Grenzprozesse. I. Math. Ann. **141** (1960), 318–342.

Motivated by iterated limiting processes, the author introduces and studies filtered summation of filters and, more generally, stacked summation of stacks. A system $\mathfrak{a}$ of subsets of a non-empty set E is called a stack (Stapel) on E, if $\mathfrak{a}$ is non-empty, $\emptyset \notin \mathfrak{a}$ and if $A \in \mathfrak{a}$ and $A \subset B \subset E$ imply $B \in \mathfrak{a}$. For a stack $\mathfrak{a}$ on E, let $\mathfrak{G}\mathfrak{a}=\{X \subset E \mid X \cap A \neq \emptyset$ for every $A \in \mathfrak{a}\}$, which is again a stack on E. For a non-empty set I of indices and a family $(K_i)_{i\in I}$ of non-empty sets, let $\mathrm{S}_{i\in I}K_i$ be the direct sum of $(K_i)_{i\in I}$, i.e., the set of all (i, k) with $i \in I$, $k \in K_i$. Let q denote the projection $q(i, k)=k$. For $X \subset \mathrm{S}_{i\in I}K_i$ and $j \in I$, let $q(X \cap (\{j\} \times K_j))$ be denoted by q_jX. Now, let $\mathfrak{a}$ be a stack on I and, for each $i \in I$, let $\mathfrak{b}_i$ be a stack on K_i. The $\mathfrak{a}$-stacked sum ${}^{\mathfrak{a}}\mathrm{S}_{i\in I}\mathfrak{b}_i$ of $(\mathfrak{b}_i)_{i\in I}$ is defined to be the stack on $\mathrm{S}_{i\in I}K_i$ formed by all $X \subset \mathrm{S}_{i\in I}K_i$ such that $q_iX \in \mathfrak{b}_i$ for $\mathfrak{a}$-almost all i (i.e., the set of those i satisfying $q_iX \in \mathfrak{b}_i$ is a member of $\mathfrak{a}$). In case $K_i=K$ and $\mathfrak{b}_i=\mathfrak{b}$ for all i, the notation $\mathfrak{a} \otimes \mathfrak{b}$ is used instead of ${}^{\mathfrak{a}}\mathrm{S}_{i\in I}\mathfrak{b}_i$. $\mathfrak{a} \otimes \mathfrak{b}$ is called the ordinal product of $\mathfrak{a}$, $\mathfrak{b}$; it is a stack on $I \times K$. The cardinal product $\mathfrak{a} \times \mathfrak{b}$ of stacks $\mathfrak{a}$, $\mathfrak{b}$ (on I, K respectively) is defined as the stack on $I \times K$ formed by all $X \subset I \times K$ such that $X \supset A \times B$ for some $A \in \mathfrak{a}$ and some $B \in \mathfrak{b}$. When $\mathfrak{a}$, $\mathfrak{b}$ are filters, $\mathfrak{a} \times \mathfrak{b}$ is the product of filters in the sense of Bourbaki. Of the various properties of these concepts, we quote the following ones. $\mathfrak{G}\,{}^{\mathfrak{a}}\mathrm{S}_{i\in I}\mathfrak{b}_i={}^{\mathfrak{G}\mathfrak{a}}\mathrm{S}_{i\in I}\mathfrak{G}\mathfrak{b}_i$, where $\mathfrak{G}$'s are taken with respect to $\mathrm{S}_{i\in I}K_i$, I and K_i respectively. ${}^{\mathfrak{a}}\mathrm{S}_{i\in I}\mathfrak{b}_i$ is a filter [ultrafilter] on $\mathrm{S}_{i\in I}K_i$, if and only if $\mathfrak{a}$ is a filter [ultrafilter] on I and $\mathfrak{b}_i$ is a filter [ultrafilter] on K_i for $\mathfrak{a}$-almost all i. In particular: $\mathfrak{G}(\mathfrak{a} \otimes \mathfrak{b})=\mathfrak{G}\mathfrak{a} \otimes \mathfrak{G}\mathfrak{b}$; $\mathfrak{a} \otimes \mathfrak{b}$ is a filter [ultrafilter] if and only if $\mathfrak{a}$, $\mathfrak{b}$ are filters [ultrafilters]. For the cardinal product, we have $\mathfrak{G}\mathfrak{a} \times \mathfrak{G}\mathfrak{b} \subset \mathfrak{G}(\mathfrak{a} \times \mathfrak{b})$, but $\mathfrak{G}\mathfrak{a} \times \mathfrak{G}\mathfrak{b}=\mathfrak{G}(\mathfrak{a} \times \mathfrak{b})$ is in general false. Also the cardinal product of two ultrafilters need not be an ultrafilter. Previously, G. Bruns and J. Schmidt [Math. Japon. **4** (1957), 133–177; MR **20** #5464] have discussed the direct sum of filters $\{\mathfrak{b}_i\}_{i\in I}$, which may be regarded as the stacked sum ${}^{\mathfrak{a}}\mathrm{S}_{i\in I}\mathfrak{b}_i$ with the special stack $\mathfrak{a}=\{I\}$. *Ky Fan* (Argonne, Ill.)

11364:

Frolík, Z. Intrinsic characterization of spaces topologically complete in the sense of E. Čech. Dokl. Akad. Nauk SSSR **137** (1961), 533–536 (Russian); translated as Soviet Math. Dokl. **2**, 323–325.

The author reformulates his characterization of absolute G_δ spaces [Czechoslovak Math. J. **10** (**85**) (1960), 359–379; MR **22** #7100] in terms of a non-negative extended real-valued function on open sets (monotone, converging to zero on the neighborhoods of a point, and such that any filter of open sets on which this function goes to zero has a cluster point). He shows that an absolute G_δ in which every additive open covering is normal admits a proper mapping ("perfect" mapping) onto a complete metric space. He gives another characterization of absolute G_δ's, by a notion related to G. Choquet's "strongly siftable" [C. R. Acad. Sci. Paris **246** (1958), 218–220; MR **19**, 1187]; where Choquet requires that every descending sequence has a common point, Frolík requires that each filter containing arbitrarily long finite descending sequences has a cluster point. {This looks like the "right" condition; but it would be desirable to know whether Choquet's notion is equivalent, stronger, weaker, or neither.} *J. R. Isbell* (Seattle, Wash.)

11365:

Armentrout, Steve. A Moore space on which every real-valued continuous function is constant. Proc. Amer. Math. Soc. **12** (1961), 106–109.

The existence of a non-constant continuous real-valued function on a nondegenerate metric space is inherent in the

definition. There are two well-known generalizations of metric spaces: Moore spaces and uniform spaces. Since uniform spaces are completely regular, non-constant functions again exist just as in the metric case. However, this is not true for Moore spaces in general, as indicated by the title.

If a Moore space is complete, it contains a dense metric subspace. While the author's space is probably connected (something he does not investigate), it is not complete. It would be interesting to have an example of a Moore space which is connected, complete and also locally connected such that every continuous real-valued function on any open subset is constant; for every such space is arcwise and locally arcwise connected. Since an arc is the homeomorphic image of [0, 1], such a space would seem to be a strange place in which so many arcs must necessarily exist. *F. B. Jones* (Chapel Hill, N.C.)

11366:

Tamano, Hisahiro. A theorem on closed mapping. Mem. Coll. Sci. Univ. Kyoto Ser A Math. **33** (1960/61), 309–315.

Let f be a closed continuous mapping of a completely regular space X onto a paracompact space Y. Theorem: X is normal if and only if, for each $y \in Y$, $f^{-1}(y)$ is normal and each bounded continuous real function on $f^{-1}(y)$ can be extended to X. Corollary: If X is paracompact and Y is normal, and the projection of $X \times Y$ onto X is closed, then $X \times Y$ is normal. {The author presumably intends his spaces to be completely regular, but does not say so. Remark 3 (p. 314) is incorrect as it stands; the case that X is infinite discrete provides a counterexample. The argument fails at the construction of the function f, where there is a confusion between X and βX. On p. 310, line 23, "$\in$" should read "$\notin$"; on p. 314, line 2, the second X should be Y.} *A. H. Stone* (Manchester)

11367:

Hewitt, Edwin. The rôle of compactness in analysis. Amer. Math. Monthly **67** (1960), 499–516.

This interesting expository paper is based on lectures given by the author as visiting lecturer for the Mathematical Association of America. The author gives convincing reasons for considering four theorems ((a) continuous real-valued functions attain their suprema, (b) the Stone-Weierstrass theorem, (c) the integral representation theorem for positive linear functionals, and (d) the ideal structure of $C(X)$) as typical of the use of compactness in analysis and he presents these theorems in some detail. A minor clarification should be made in the author's statement of (b): he considers the constant functions to be polynomials in f for any function f.

J. L. Kelley (Berkeley, Calif.)

11368:

Nakano, Hidegorô. On compactness of weak topologies. Proc. Japan Acad. **35** (1959), 444–445.

If for each m in an index set M there is given a map f_m of a set R into a topological space S_m, then the weak topology for R is the weakest (coarsest, smallest) which makes all f_m continuous. Equivalently, if f is on R to $\times\{S_m : m \in M\}$ is such that $f(r)_m = f_m(r)$, then the weak topology consists of inverses under f of sets open in the product topology. Continuing earlier work, the author seeks criteria for weak compactness in case all S_m are compact but not necessarily Hausdorff. He gives as sufficient condition: if $s \in f[R]^-$, then $f(x) \in \{s\}^-$ for some x in R. The condition is not necessary, and he explores a variant which is necessary but not sufficient. It may be worthwhile to note in this connection that a necessary and sufficient condition for compactness of a subset X of a compact space Y is that $T[X] = \{y : (x, y) \in T$ for some $x \in X\}$ be closed, where T is the closure of the diagonal in $Y \times Y$. *J. L. Kelley* (Berkeley, Calif.)

11369:

Tamano, Hisahiro. A note on the pseudo-compactness of the product of two spaces. Mem. Coll. Sci. Univ. Kyoto Ser. A Math. **33** (1960/61), 225–230.

Let X and Y be infinite completely regular Hausdorff spaces. Glicksberg [Trans. Amer. Math. Soc. **90** (1959), 369–382; MR **21** #4405] showed that $\beta(X \times Y) = \beta X \times \beta Y$ if and only if $X \times Y$ is pseudocompact (a condition known to require more than pseudocompactness of X and Y). In this note it is proved that the following statements are equivalent. (1) $\beta(X \times Y) = \beta X \times \beta Y$. (2) The tensor product $C^*(X) \otimes C^*(Y)$ is dense in $C^*(X \times Y)$. (3) Both X and Y are pseudocompact, and the projection of $Z(F)$ into X is closed for each $F \in C^*(X \times Y)$.

By applying this theorem and Glicksberg's result, the author shows that if X and Y are pseudocompact, and if every non-P-point $x \in X$ has the property that whenever x is a limit point of a subset H of X, there is a compact subset C of X such that x is a limit point of $C \cap H$, then $X \times Y$ is pseudocompact. *C. W. Kohls* (Rochester, N.Y.)

11370:

Stone, A. H. Hereditarily compact spaces. Amer. J. Math. **82** (1960), 900–916.

The author treats topological spaces for which each subspace is compact. The interesting spaces of this type are not T_2, and consequently compact means that each open cover has a finite subcover (without any separation axioms assumed). Such spaces have been studied in connection with algebraic constructions.

First, some characterizations of these spaces are given, most of which apply in general only when X is T_1 (which is the case in the algebraic studies). Next, an ordinal number (called the type of the space) is associated with each such space so that it reflects, in the finite case and, loosely speaking, in the infinite case, the greatest possible length of a collection of strictly decreasing closed subsets. The type is then studied in detail. A modified version of the direct limit shows how the study of hereditarily compact spaces of arbitrary type α can be reduced to the study of such spaces of types smaller than α.

H. H. Corson (Seattle, Wash.)

11371:

Freudenthal, H. Bündige Räume. Fund. Math. **48** (1959/60), 307–312.

This is a postscript to a previous paper in Fund. Math. **39** (1952), 189–210 [MR **14**, 893]. The author notes a minor inaccuracy, which occurs already earlier [Ann. of Math. (2) **43** (1942), 261–279; MR **3**, 315] but which is really immaterial, and he points out simplifications to be effected by various modifications of detail.

L. C. Young (Madison, Wis.)

11372:

Nagami, Keiô. Finite-to-one closed mappings and dimension. III. Proc. Japan Acad. **36** (1960), 405–407.

[For part II see same Proc. **35** (1959), 437–439; MR **22** #4052.] Theorems are stated giving necessary and sufficient conditions, in terms of the existence of suitable sequences of open coverings $\mathscr{U}_i$ ($i=1, 2, \cdots$), for a topological space to be metrizable and of dimension $\leqq n$. One of them generalizes a theorem of Dowker and Hurewicz [Fund. Math. **43** (1956), 83–88; MR **18**, 56]; the others generalize theorems of J. Nagata [Fund. Math. **45** (1958), 143–181; MR **21** #3827]. The main feature of these generalizations is that the usual condition "order of each $\mathscr{U}_i \leqq n+1$" is weakened to "for each point x, lim inf (order of $\mathscr{U}_i$ at x) $\leqq n+1$". It is also observed that the completion of an n-dimensional metric space, in a characteristic metric due to Nagata, is n-dimensional; and this result is generalized. No proofs are given, but it is stated that they use the methods of part I of this series [Proc. Japan Acad. **34** (1958), 503–506; MR **21** #862]. *A. H. Stone* (Manchester)

11373:

Jaroń, J. On the extensibility of mappings, their local properties and some of their connections with the dimension theory. Fund. Math. **48** (1959/60), 287–305.

Recall Kuratowski's notation $X\tau Y$, which means that for every closed $F \subset X$ the restriction map $Y^X \to Y^F$ is surjective. The author localizes the relation τ, in a manner reminiscent of dimension theory, as follows: $X\varphi Y$ if each $x \in X$ has arbitrarily small neighborhoods U_x with $\dot{U}_x\tau Y$. Some immediate, purely formal, properties of φ are established, as well as the following results relating τ and φ: Let X, Y, be separable metric, Y an ANR, and $X\varphi Y$; then (i) $X=A \cup B$, where $A \cap B = \emptyset$, A is an F_σ with $A\tau Y$, and dim $B \leqq 0$, (ii) each pair of disjoint closed sets in X can be separated by some closed F with $F\tau Y$, (iii) if X is compact and not $X\tau Y$, there is a closed $F \subset X$ such that no subset $A \subset F$ with $A \times I\tau Y$ separates F (F is called a φ-manifold rel Y in analogy to Cantor manifolds, which are, in this terminology, φ-manifolds rel spheres). *J. Dugundji* (Los Angeles, Calif.)

11374:

Maxwell, C. N. An order relation among topological spaces. Trans. Amer. Math. Soc. **99** (1961), 201–204.

The author defines a partial ordering on a class of topological spaces as follows: For two spaces X and Y in the class, $X \leqq Y$ if for each open cover α of X there is a continuous function $f: X \to Y$ which is an onto α-map (i.e., there is an open cover β of Y such that $f^{-1}(\beta)$ refines α). Properties of a space Y which are inherited by a space X satisfying $X \leqq Y$ are considered first. Then, under the hypothesis that X and Y are (metric) ANR and $X \leqq Y$, the following results are obtained: (1) for each open cover α of X there is an onto α-map $f: X \to Y$ and a $g: Y \to X$ such that gf is α-homotopic to the identity on X and (2) if Y has the fixed-point property, then X also has it. These theorems are generalizations of results of Eilenberg [Fund. Math. **30** (1938), 92–95] and Borsuk [Fund. Math. **31** (1938), 154–159], respectively, in which the spaces are required to be compact ANR. The examples given need modification in order to be ε-mappings. *J. Segal* (Seattle, Wash.)

11375:

Edelstein, Michael. An extension of Banach's contraction principle. Proc. Amer. Math. Soc. **12** (1961), 7–10.

A mapping f of a complete metric space X into itself has a unique fixed point, if there exist $\varepsilon > 0$, $0 < \alpha < 1$ satisfying the following conditions: (1) For any two points a, b of X, there is a finite chain $a = x_0, x_1, \cdots, x_n = b$ of points of X such that $d(x_{i-1}, x_i) < \varepsilon$ for $i = 1, 2, \cdots, n$; (2) $d(x, z) < \varepsilon$ and $d(y, z) < \varepsilon$ imply $d(f(x), f(y)) < \alpha d(x, y)$. *Ky Fan* (Argonne, Ill.)

11376:

Reichbach, Marian. A note on polynomial mappings in uniform spaces. Riveon Lematematika **13** (1959), 26–28.

The well-known argument (open-closed map to a connected space) for showing a map is onto is shown to hold for the following situation: A continuous map $F: X \to Y$, where X and Y are both metric spaces, is a polynomial map if, for every sequence $\{x_n\}$, inf $d(x_n, x_m) > 0$ implies inf $d(Fx_n, Fx_m) > 0$. Let the vicinities of the diagonal of X have a countable basis, where X is a complete linear topological space. Let F be a polynomial self-map of X such that for each $y_0 \in X$ there is a point $x_0 \in F^{-1}(y_0)$, a neighborhood $U(y_0)$ and a complex nonzero constant λ dependent on y_0 for which the iterates of $Tx = x - \lambda(F(x) - y)$, $y \in U(y_0)$ and x possibly dependent on y, constitute a Cauchy sequence. Then F is onto. *D. G. Bourgin* (Urbana, Ill.)

11377:

Granas, A.; Jaworowski, J. W. Some theorems on multi-valued mappings of subsets of the Euclidean space. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 277–283. (Russian summary, unbound insert)

The Vietoris theorem is applied to multivalued acyclic upper semi-continuous compactness-preserving maps in Euclidean space. Denote such a map by f. The main theorem is that if $f: E^{n+1} \to E^{n+1}$ has the additional property that $f(x_1) \cap f(x_2) \neq \emptyset$ implies $d(x_1, x_2) < \varepsilon$ then, in generalization of a theorem of Borsuk, $f(E^{n+1})$ is an open set. (It should be stated that the extensions to multivalued maps of the index and homotopy notions when images are acyclic compact have been essentially known to many workers in these fields even in the absence of an explicit statement in the literature.) *D. G. Bourgin* (Urbana, Ill.)

11378:

Kister, J. M. Isotopies in 3-manifolds. Trans. Amer. Math. Soc. **97** (1960), 213–224.

An isotopy H_t, $t \in I = [0, 1]$, of a space M with metric ρ, is an ε-isotopy ($\varepsilon > 0$) if $\rho(H_{t_1}(x), H_{t_2}(x)) < \varepsilon$ for each $x \in M$ and t_1, t_2 in I. The main result of the paper is the following theorem. Suppose M is a 3-manifold with boundary having triangulation Σ, and ρ is the natural metric for Σ. Then for any $\varepsilon > 0$ there is a $\delta > 0$ so that if f and g are homeomorphisms of M onto itself, and $\rho(f(x), g(x)) < \delta$ for all x in M, there is an ε-isotopy of M taking g onto f. One corollary is this same theorem with M a 2-manifold. Another characterizes, for a compact 3-manifold with boundary, the homeomorphisms isotopic to the identity. The proof of the theorem, previously outlined in a research announcement [Bull. Amer. Math. Soc. **65**

(1959), 371–373; MR **21** #5957], is given in detail. It employs a number of lemmas of intrinsic interest, largely pertaining to local connectedness in the space of self-homeomorphisms of M. There are references to certain related results by others, including the case $M=E^3$ of the theorem, the case M compact, and an announced generalization asserting that the space of homeomorphisms of M onto itself, under the compact-open topology, is locally n-connected for each n. *S. S. Cairns* (Urbana, Ill.)

11379:

Mann, L. N. Compact abelian transformation groups. Trans. Amer. Math. Soc. **99** (1961), 41–59.

Let G be a toral group acting on a space X. It was shown by Floyd [Ann. of Math. (2) **66** (1957), 30–35; MR **19**, 571] that if X is a compact manifold which is a homology sphere over R, the reals mod 1, then the fixed-point set $F(X)$ is a homology sphere over R. The present author shows that the manifold condition can be replaced by the purely homological condition that X be a homological manifold (suitably defined) over R. In fact, if X is a homological manifold over R, so is each component of $F(X)$, and if X is orientable, so are the components. If, in addition, X is a compact homology sphere over R, so is $F(X)$. In the course of the proof the author extends to locally orientable homological manifolds over R the result of Floyd [ibid. **65** (1957), 505–512; MR **19**, 292] that there exist only a finite number of distinct isotropy groups G_x when x is restricted to a compact subset of X. *P. A. Smith* (New York)

11380:

Hahn, F. J. Nets and recurrence in transformation groups. Trans. Amer. Math. Soc. **99** (1961), 193–200.

The author offers a suggestive alternative approach to the notion of recurrent point in transformation groups which is equivalent to that in Gottschalk and Hedlund, *Topological dynamics* [Amer. Math. Soc. Colloq. Publ., Vol. 36, Amer. Math. Soc., Providence, R.I., 1955; MR **17**, 650]. Let (X, T, π) be a transformation group, where T is an abelian group. The author associates certain kinds of orderings of T with semigroups and replete semigroups in T. As a consequence he obtains a characterization of a recurrent point $x\in X$ in terms of a convergence property of the motion π_x. He shows how the technique of nets and convergence provides proofs of various theorems on recurrence, for example, the inheritance theorem for recurrent points. *W. H. Gottschalk* (New Haven, Conn.)

11381:

Baum, John D. An equicontinuity condition in topological dynamics. Proc. Amer. Math. Soc. **12** (1961), 30–32.

Let T be a topological group. A subset S of T is said to be replete in T, provided that S contains some bilateral translate of each compact subset of T. The author proves the following theorem. Let X be a compact Hausdorff space, T an abelian topological group, and (X, T, π) a transformation group; then the following statements are equivalent. (I) For each index α of X there exist a replete semigroup $P_{\alpha\beta}\subset T$ and an index β of X such that $(x, y)\in\beta$ implies $(xp, yp)\in\alpha$ for each $p\in P_{\alpha\beta}$. (II) T is uniformly equicontinuous.

For a generalization of the above result where the conditions that T be abelian and the $P_{\alpha\beta}$ be semigroups are removed, see Jesse Clay, Thesis [University of Pennsylvania, 1961]. *R. Ellis* (Philadelphia, Pa.)

11382:

Bryant, B. F. On expansive homeomorphisms. Pacific J. Math. **10** (1960), 1163–1167.

Let X be a compact separated uniform space, let φ be a homeomorphism of X onto X, and let φ be expansive, that is, there exists an index α of X such that if $x, y\in X$ with $x\neq y$, then there exists an integer n such that $(x\varphi^n, y\varphi^n)\notin\alpha$. The author proves the following theorems. (1) X is metrizable. (2) If X is infinite, then there exist $a, b, c, d\in X$ such that a and b are positively asymptotic under φ and such that c and d are negatively asymptotic under φ. (3) If X is self-dense, if $x\in X$, and if U is a neighborhood of x, then there exists $y\in U$ such that x and y are not doubly asymptotic under φ. Theorem 2 (for metric spaces) was first proved by S. Schwartzman in his dissertation [Yale University, 1952].

W. H. Gottschalk (New Haven, Conn.)

ALGEBRAIC TOPOLOGY

See also 11065, 11412.

11383:

Boltyanskiĭ, V. G.; Postnikov, M. M. On the fundamental concepts of algebraic topology. Axiomatic definition of cohomology groups. Dokl. Akad. Nauk SSSR **133** (1960), 745–747 (Russian); translated as Soviet Math. Dokl. **1**, 900–902.

In this note the authors characterize axiomatically the absolute reduced cohomology groups. The advantages claimed for this axiomatization are (i) the simplicity consequent on avoiding mention of relative groups and (ii) the complete duality with an axiomatization of homotopy groups. The duality is in the sense of Eckmann and Hilton [C. R. Acad. Sci. Paris **246** (1958), 2444–2447, 2555–2558, 2991–2993; MR **20** #6694, 6695, 6696] and the basic notion used in the characterization is that of a cofibration as defined in the papers quoted above. Omitting a few category-theoretical technicalities, the axioms, relating to contravariant functors H^n, $n\geqq 0$, defined on a category of based spaces and passing to abelian groups, are as follows: (1) $H^n(S^0)$ is trivial for $n>0$; (2) if $f\simeq g\colon X\to Y$, then $H^n(f)=H^n(g)\colon H^n(Y)\to H^n(X)$; (3) if $X\xrightarrow{f}Y\xrightarrow{g}Z\to *$ is a cofibration sequence, then the sequence

$$\cdots\to H^n(Z)\xrightarrow{H^n(g)}H^n(Y)\xrightarrow{H^n(f)}H^n(X)\xrightarrow{\partial^n}H^{n+1}(Z)\to\cdots$$

is exact, where ∂^n is a natural transformation. The group $H^0(S^0)$ is called the coefficient group. It is easy to see that the singular cohomology groups satisfy the axioms and that the axioms uniquely determine the cohomology groups of compact polyhedra with a given coefficient group.

The note ends with the provocative claim that homology does not play so primary a role as cohomology. {The reviewer endorses this claim provided that axiomatics, rather than, say, geometrical concepts, are in question.}

P. J. Hilton (Birmingham)

11384:

Kawada, Yukiyosi. Cosheaves. Proc. Japan Acad. **36** (1960), 81–85.

The author sketches outlines of a homology theory with coefficients in a "cosheaf" (as a pendant to cohomology with coefficients in a sheaf). Very similar theories were developed recently by E. Luft [Bonn. Math. Schr. no. 8 (1959); MR **21** #3841] and R. Kultze [Arch. Math. **10** (1959), 438–442; MR **22** #5030]. In this paper a pre-cosheaf $\mathfrak{F}$ on a paracompact space X is a covariant functor defined on the category $\mathfrak{A}(X)$ of closed subsets $A \subset X$ and inclusion maps; $\mathfrak{F}$ takes its values in the category of compact Abelian groups and homomorphisms $\iota_{A,B}$: $\mathfrak{F}(B) \to \mathfrak{F}(A)$, $A \supset B$. (Compactness seems to be the specific feature distinguishing the theory from that of E. Luft.) With every $A \in \mathfrak{A}(X)$ is considered the family of all $B \in \mathfrak{A}(X)$ for which $A \subset \operatorname{Int} B$; it is required that the corresponding inverse system of groups $\mathfrak{F}(B)$ has $\mathfrak{F}(A)$ as its inverse limit. Cosheaves are defined as pre-cosheaves having the following property. If $\{A_1, \cdots, A_n\}$ is a closed covering of $A \in \mathfrak{A}(X)$, then $\mathfrak{F}(A) = \sum_i \iota_{A,A_i}\mathfrak{F}(A_i)$, and if $\sum_i \iota_{A,A_i}s_i = 0$, $s_i \in \mathfrak{F}(A_i)$, then there exist $s_{ij} \in \mathfrak{F}(A_i \cap A_j)$ such that $s_{ii} = 0$, $s_{ij} = -s_{ji}$ and $s_i = \sum_j \iota_{A_i, A_i \cap A_j} s_{ij}$.

The author proceeds by defining (for compact X) complete pre-cosheaves. $\mathfrak{F}$ is complete provided all $\iota_{A,B}$, A, $B \in \mathfrak{A}(X)$, are monomorphisms. With every cosheaf is canonically associated a complete cosheaf $\tilde{\mathfrak{F}}$. This enables the author to produce complete resolutions of cosheaves, by means of which he then defines homology groups $H_n(X, \mathfrak{F})$ with coefficients in the cosheaf $\mathfrak{F}$. The compact homology groups thus obtained are uniquely determined by a set of five axioms, similar to the axioms of E. Luft (with no restrictions in the exactness axiom). Same homology groups are also obtainable by the Čech procedure. Finally, with every sheaf $\mathfrak{S}$ of discrete groups is associated a cosheaf of compact character groups—the dual cosheaf $\mathfrak{F}$. A theorem asserts that $H_n(X, \mathfrak{F})$ is the character group of $H^n(X, \mathfrak{S})$. *S. Mardešić* (Zagreb)

11385:

Bauer, Friedrich-Wilhelm. Zur Dimensionstheorie der Kompakten im R^n. Math. Ann. **131** (1956), 393–410.

Let F be a compact subset of Euclidean n-space R^n. A real-valued non-negative function f_q defined over the space of all closed sets of F is called a dimensional function if (1) $f_q(M) = 0 \rightleftarrows \dim M \leq q$, (2) $\limsup f_q(M_i) \leq f_q(\lim M_i)$ for any convergent sequence $\{M_i\}$ and (3) f_q is monotone. The main result of this paper reads as follows: Let $\dim F = r$ and let f_q be any dimensional function ($q = -1, 0, 1, \cdots$); then (1) for any polyhedral chain c^p with $|\partial c^p| \cap F = 0$ and for any $\varepsilon > 0$ there exists a chain $c_1^p \sim c^p$ in $U(|c^p|, \varepsilon)$ with $f_{r+p-n}(|c_1^p| \cap F) < \varepsilon$, and (2) there exist a polyhedral chain c^p with $|\partial c^p| \cap F = 0$ and a positive number γ such that $f_{r+p-n-1}(|c_1^p| \cap F) > \gamma$ for every $c_1^p \sim c^p$ in $U(|c^p|, \gamma)$. The author deduces also the well-known "Rechtfertigungssatz" of P. Alexandroff [see Uspehi Mat. Nauk **6** (1951), no. 5 (45), 43–68; MR **13**, 764] that $\dim F \leq r$ ($r \leq n-1$) if and only if for $p < n-r-1$ every cycle z^p in $R^n - F$ which is bounding in $U(|z^p|, \varepsilon)$ is bounding also in $U(|z^p|, \varepsilon) - F$, and there exists a positive number γ such that for every $\varepsilon > 0$ there exists a cycle z^{n-r-1} in $R^n - F$ which is bounding in $U(|z^{n-r-1}|, \varepsilon)$ but not in $U(|z^{n-r-1}|, \gamma) - F$. Here the group of integers is taken as the coefficient domain of chains.

K. Morita (Zbl **71**, 160)

11386:

Ruhadze, L. R. On local duality theorems. Dokl. Akad. Nauk SSSR **131** (1960), 1257–1260 (Russian); translated as Soviet Math. Dokl. **1**, 422–425.

The main result is a local analogue of the Alexander-Pontryagin duality theorem. Prototypes for this result are due to Čech [Ann. of Math. (2) **35** (1934), 678–701] and Alexandroff [ibid. **36** (1935), 1–35]. In particular, let A be closed in S^n, $x \in A$, $\{U_\lambda\}$ a family of spherical neighborhoods of x, $\{F_\alpha\}$ the set of all compact subsets of A, C a compact coefficient group, D a discrete group dual to C. Then the homology groups $\lim_\lambda \lim_\alpha H^r(F_\alpha, F_\alpha - U_\lambda; C)$ and the Betti groups $\lim_\lambda \lim_\alpha B^{n-r-1}(U_\lambda - F_\alpha; D)$ are dual, with the multiplication defined by linking.

T. R. Brahana (Athens, Ga.)

11387:

Raymond, Frank. The end point compactification of manifolds. Pacific J. Math. **10** (1960), 947–963.

Let S always denote a locally compact Hausdorff space, and let $A(S)$ denote the family of all open subsets of S with compact closure. Let X denote an S which is also connected and locally connected. Using an inverse limit process, the author constructs and then characterises neatly the (Freudenthal) end-point compactification X' of X, i.e., he embeds X in X' so that X is open, $B = X' - X$ is totally disconnected, and if Q is open connected in X', then $Q - B$ is connected. If B contains exactly k points, then X has exactly k ends in the sense of Specker, and conversely. Next, with coefficients in a principal ideal domain L, there is a natural map

$$H^*(B) = \operatorname{Dlim} H^*(S' - U) \to \operatorname{Dlim} H^*(S - U) = I^*(B) \qquad (U \in A(S))$$

which is an isomorphism if S' is a polyhedral manifold with (manifold) boundary B. Analogously, he defines groups $I_p(B)$, but using the homology groups of Borel and Moore [Michigan Math. J. **7** (1960), 137–159], to which Poincaré duality is applicable. His main theorem is then the following. Let X be an orientable n-gm ($n > 1$) over L. Then X' (the end-point compactification of X) is an orientable n-gm if and only if $I_p(X) = 0$ ($p \neq 1$, n), and $H^*(X')$ is finitely generated. The main corollary is that if the compact cohomology of X is that of Euclidean n-space, then the one-point compactification is an orientable n-gm with cohomology isomorphic to that of the n-sphere. There is an interesting application to 2-manifolds: X is the complement, in a closed 2-manifold, of a closed totally disconnected subset if and only if $H^1(X')$ is finitely generated (using integers mod 2 if X is non-orientable, and rationals otherwise). Finally, suppose X is a compact connected orientable n-gm with boundary B. Let $f: X \to X^*$ be the map which collapses each component B_i of B to a point x_i. Then X^* is an orientable n-gm if and only if each B_i is an orientable spherelike $(n-1)$-gm.

H. B. Griffiths (Birmingham)

11388:

Raymond, Frank. Separation and union theorems for generalized manifolds with boundary. Michigan Math. J. **7** (1960), 7–21.

The author refines theorems of Wilder and of White concerning the separation of an n-gm by an $(n-1)$-gm, and the converse problem. They are refinements because they hold for generalised manifolds (gm's) over a principal ideal domain L, which are locally compact Hausdorff. Sample theorems are: (1) Let X_1, X_2 be closed subsets of $X=X_1\cup X_2$ which are locally orientable (l.o.) n-gm's with l.o. boundaries B_1, B_2. Suppose that $X_1\cap X_2=B\subseteq B_1\cap B_2$ is a l.o. $(n-1)$-gm. Then X is a l.o. n-gm with l.o. boundary $(B_1\cup B_2)-(X_1\cap X_2)$. (2) If the connected l.o. n-gm X is separated by a l.o. connected $(n-1)$-gm X', then $X-X'$ has exactly two components A_i, each with frontier X', and such that $A_i\cup X'$ is a manifold with boundary.

Applying these results, he proves his Theorem 6: Let $C=A\times B$ be a l.o. n-gm with l.o. boundary C' over a field or over the integers. Then A [resp. B] is a l.o. r_0-gm [resp. s_0-gm] with possible l.o. boundary A' [resp. B'], where $r_0+s_0=n$. (The converse, for a field, was proved by Brahana [Illinois J. Math. **2** (1958), 76–80; MR **20** #2720].) It is important in setting a limit to recent theorems where C is a classical manifold but A is not [see, e.g., Bing, Bull. Amer. Math. Soc. **64** (1958), 82–84; MR **20** #3514]. *H. B. Griffiths* (Birmingham)

11389:

McMillan, D. R., Jr. On homologically trivial 3-manifolds. Trans. Amer. Math. Soc. **98** (1961), 350–367.

The composition of two oriented 3-manifolds M and N is the oriented 3-manifold obtained by excising a nicely placed small open 3-ball from M and similarly from N, and then identifying the boundary 2-spheres thus created. A regular free manifold is the composition of a number of solid tori of genus one and a number of closed 3-balls. Let M be a compact 3-manifold without boundary, such that $H_1(M;Z)=0$, and such that each polyhedral simple closed curve is homologous to zero mod 2 in some regular free manifold which is a subspace of M. The main result of the paper is that M is a 3-sphere. An example shows the necessity of the condition $H_1(M;Z)=0$. This paper extends the work of R. H. Bing [Ann. of Math. (2) **68** (1958), 17–37; MR **20** #1973] in characterizing the 3-sphere by a combination of its algebraic and geometric properties. *D. Epstein* (Princeton, N.J.)

11390:

Curtis, M. L. Homotopically homogeneous polyhedra. Michigan Math. J. **8** (1961), 55–60.

Let X denote a space and $D(X)$ the diagonal in $X\times X$. Furthermore, let $\hat{X}=X\times X-D(X)$ and let $p:\hat{X}\to X$ be the map defined by the natural projection $p(x_1,x_2)=x_1$. If $(\hat{X},X,p)$ is a Hurewicz fiber space [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 956–961; MR **17**, 519], X is said to be homotopically homogeneous (h.h.). Examples of h.h. spaces are manifolds, topological groups and loop spaces. If N is a closed neighborhood of a point x in X and $\dot{N}$ is its boundary, then N is a conical neighborhood of x if there exists a homeomorphism of N onto the cone $C\dot{N}$ of $\dot{N}$ which is the identity on $\dot{N}$ and sends x to the vertex of the cone. The space X is locally conical if each point of X has a conical neighborhood. A point x of a space X is called an r-point if x has arbitrarily small (closed) neighborhoods U such that for each $y\in U$, there exists a (strong) deformation retraction of $U-y$ onto $\dot{U}$ [A. Kosiński, Fund. Math. **42** (1955), 111–124; MR **17**, 654].

Theorem: A locally conical point of an h.h. space is an r-point. Corollary: An h.h. polyhedron is a homotopy manifold.

According to the author, no example is known of an h.h. polyhedron which is not a manifold. H.h. polyhedra of dimensions 1, 2, 3 are manifolds, and it is further shown that 4-dimensional h.h. polyhedra are manifolds if the Poincaré conjecture is true. The paper also contains an alternate proof of a result of Kwun and Raymond [Proc. Amer. Math. Soc. **11** (1960), 135–139; MR **22** #7111] that a locally conical 3-gcm (generalized closed manifold) is a manifold. *E. Fadell* (Madison, Wis.)

11391:

Curtis, M. L. Shrinking continua in 3-space. Proc. Cambridge Philos. Soc. **57** (1961), 432–433.

First, the definition of a homotopy manifold is slightly strengthened. Then the following result is obtained. Let C be a continuum in E^3, and let X be the quotient space obtained by collapsing C to a point. If X is a homotopy manifold, then X is homeomorphic to E^3.

The proof makes use of Whitehead's version [Bull. Amer. Math. Soc. **64** (1958), 161–166; MR **21** #2241] of the sphere theorem. Also, remarks are made regarding a generalization of the above result and an analogous result. *K. W. Kwun* (Ann Arbor, Mich.)

11392a:

Deheuvels, René. Espaces fibrés. Séminaire P. Lelong, 1957/58, exp. 7, 15 pp. Faculté des Sciences de Paris, 1959.

11392b:

Deheuvels, René. Espaces fibrés. Bull. Soc. Math. Belg. **10** (1958), 4–18.

[Les deux articles ont le même texte.]

L'auteur étudie des généralisations intéressantes de la notion d'espace fibré à faisceau structural. Il reformule ainsi une partie des résultats démontrés deux ans plus tôt par le rapporteur et dont il ne semble pas avoir connaissance [Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 270–290; MR **19**, 973]. Sa terminologie est souvent peu conformiste: par exemple il utilise l'expression "isomorphisme local" pour désigner, non ce que l'on appelle d'habitude ainsi, mais ce qui est universellement dénomé "atlas". La loi de composition entre structures fibrées introduite par l'auteur n'est autre que celle du groupoïde de cohomologie de dimension un à coefficients dans un système convenable défini par le rapporteur [loc. cit.; voir aussi Symposium internacional de topología algebraica, pp. 309–322, Universidad Nacional Autónoma de México, Mexico City, 1958; Canad. J. Math. **12** (1960), 231–251; MR **20** #4828; **22** #1888]. *P. Dedecker* (Liège)

11393:

Peterson, Franklin P. A note on H-spaces. Bol. Soc. Mat. Mexicana (2) **4** (1959), 30–31.

It is known that, if $\phi_i:(X\times X,(x_0,x_0))\to(X,x_0)$ $(i=1,2)$ are two H-structures on an H-space X, and if $\Omega\phi_i(f,g)(t)=\phi_i(f(t),g(t))$ are the induced H-structures on the loop

space $(\Omega X, x_0)$, the structures $\Omega\phi_i$ and the H-structure defined by the multiplication of loops are homotopy-commutative and homotopy-equivalent to one another.

The author shows by an example that a similar result does not hold for different H-spaces X_i $(i=1, 2)$ where ϕ_i is a multiplication on X_i: even when these H-structures are homotopy-commutative and ΩX_1 is of the same homotopy type as ΩX_2, $\Omega\phi_1$ and $\Omega\phi_2$ are not homotopy-equivalent.

The spaces X_i are fibre spaces over the Eilenberg-MacLane space $K(Z, 3)$ with $K(Z, 5)$ as fibre; X_1 is the cartesian product and X_2 has as k-invariant the square of the fundamental class; the author shows that $H^4(\Omega X_2; Z_2)$ contains non-primitive elements, whereas in ΩX_1 every element of $H^4(\Omega X_1; Z_2)$ is primitive.

G. Hirsch (Brussels)

11394:

Sugawara, Masahiro. On the homotopy-commutativity of groups and loop spaces. Mem. Coll. Sci. Univ. Kyoto Ser. A Math. **33** (1960/61), 257–269.

The author introduces a strengthened notion of homotopy-commutativity of an H-space and then proves the following two theorems. (1) A countable CW-complex B is an H-space if and only if $\Omega(B)$ is strongly homotopy-commutative. (2) If G is a countable CW-group and if B_G is a countable CW-complex which is a classifying space for G, then B_G is an H-space if and only if G is strongly homotopy-commutative.

N. Stein (New York)

11395:

Browder, William. Homology operations and loop spaces. Illinois J. Math. **4** (1960), 347–357.

The concept of an H_n-space and the so-called H-squaring operations in an H_n-space were introduced by T. Kudo and S. Araki [Mem. Fac. Sci. Kyūsyū Univ. Ser. A. **10** (1956), 85–120; MR **19**, 442]. The H-squaring operations are homology operations on one variable which may be regarded as duals of the Steenrod squares. Kudo and Araki used these operations to determine the homology ring with mod 2 coefficients of the iterated loop space on a sphere. In the present paper, the author reformulates the definitions of Kudo and Araki in a briefer and more conceptual form, and gives a new treatment of their H-squaring operations. He also introduces a new homology operation on two variables defined for any coefficient domain. Finally, he applies his methods to give an explicit determination of the homology ring of the iterated loop space of the iterated suspension of a space mod 2, in terms of the homology of the original space mod 2. The mod 2 cohomology ring is also determined. This result may be regarded as a generalization of the result of Kudo and Araki on the iterated loop space of a sphere.

W. S. Massey (New Haven, Conn.)

11396:

Barratt, M. G. Spaces of finite characteristic. Quart. J. Math. Oxford Ser. (2) **11** (1960), 124–136.

The characteristic of a complex A is the order of the identity map of EA, the suspension of A, under track addition. In studying the homotopy groups of a space of finite characteristic it suffices to restrict attention to those of prime-power characteristic. Suppose p is a prime and A is of characteristic p^m and $(n-1)$-connected $(n\geqq 1)$. Then $p^{mk}\pi_q(EA)=0$ if $q\leqq 2^k n$. If A is a suspension, then $p^{m+k}\pi_q(EA)=0$ if $q\leqq p^{k+1}n$.

N. Stein (New York)

11397:

Neuwirth, Lee. The algebraic determination of the genus of knots. Amer. J. Math. **82** (1960), 791–798.

The principal result is a description of the commutator subgroup $[G, G]$ of the group $G=\pi_1(S^3-k)$ of a tame knot k: Either $[G, G]$ is free of rank $2g$, where g is the genus of k, or it is a certain special kind of infinitely generated free product with amalgamation, or it is a locally free direct limit of free groups of rank $2g$. From this a number of corollaries are deduced. Corollary 1: If $[G, G]$ is finitely generated and $S\subset S^3$ is an orientable surface of minimal genus g spanning k, then $\pi_1(S^3-S)$ is not free. Corollary 2: The center of $[G, G]$ is trivial. Corollary 3: If k is non-trivial then G has a subgroup that is free of rank 2. Corollary 4: The commutator series of G does not terminate after a finite number of steps unless k is trivial. Corollary 5: The center Z of G is cyclic. Theorem 2: If Z is non-trivial, it intersects non-trivially every abelian subgroup of G of rank 2. Corollary 6: If Z is non-trivial, then $[G, G]$ has no abelian subgroups of rank 2. Corollary 7: If $[G, G]$ is finitely generated, then the degree δ of the Alexander polynomial is equal to $2g$. (It is known that $\delta\leqq 2g$ for any tame knot [H. Seifert, Math. Ann. **110** (1934), 571–592], and that $\delta=2g$ if k is an alternating knot [K. Murasugi, J. Math. Soc. Japan **10** (1958), 235–248; MR **20** #6103a; R. Crowell, Ann. of Math. (2) **69** (1959), 258–275; MR **20** #6103b].) There are alternating knots (e.g., 5_2) for which $[G, G]$ is not finitely generated, and there are non-alternating knots (e.g., the torus knot of type 3, 5) for which $[G, G]$ is finitely generated. From Corollary 6 and some results of the author that have not yet been published it follows that if k is alternating then k is a torus knot if and only if Z is non-trivial; it is conjectured that this is also true for non-alternating knots k.

R. H. Fox (Princeton, N.J.)

11398:

Dirac, G. A.; Stojaković, M. D. ★Problem četiri boje [The four-color problem]. Matematička Biblioteka, 16. Univerzitet u Beogradu, Belgrade, 1960. ii+63 pp. $1.50.

The booklet is a stimulating, popular exposition of the four-color problem and related map and graph coloring questions. Most of the proofs are given; for others the reader is referred to the original papers. The book contains many illustrations and is very readable. The only serious error found is on p. 43, where it is "proved" that the four-color conjecture is not an undecidable proposition.

B. Grünbaum (Seattle, Wash.)

DIFFERENTIAL TOPOLOGY

11399:

Morse, Marston. The existence of non-degenerate functions on a compact differentiable m-manifold M. Ann. Mat. Pura Appl. (4) **49** (1960), 117–128.

This paper brings an elementary and essentially self-contained account of the following useful folk-theorem.

Let $M \subset E_n$ be a smooth submanifold of Euclidean n-space. Then except for a set of measure zero the linear functions on E_n restrict to nondegenerate functions on M. If in addition M is compact, then the linear functions with nondegenerate critical points on M form an open set.

The proof is based on Sard's theorem applied to the map which takes the normal bundle to M into E_n, by mapping the pair (p, v), $p \in M$, $v \in E_n$, v perpendicular to M at p, into v. *R. Bott* (Cambridge, Mass.)

11400:

Stewart, T. E. On groups of diffeomorphisms. Proc. Amer. Math. Soc. **11** (1960), 559–563.

Let first K be the space of all diffeomorphisms of Euclidean n-space with the C^r topology. Then it is proved that the orthogonal group O_n is a deformation retract of K. Next let K be the space of diffeomorphisms of the n-sphere with the C^r topology. Then the inclusion O_{n+1} into K induces a monomorphism of the homotopy groups. *S. Smale* (Berkeley, Calif.)

11401:

Palais, Richard S.; Stewart, Thomas E. Deformations of compact differentiable transformation groups. Amer. J. Math. **82** (1960), 935–937.

Let $\varphi: G \times M \to M$ be a differentiable action of a Lie group G on a differentiable manifold M. A deformation of φ is a homotopy φ_t of differentiable actions of G on M such that $\varphi_0 = \varphi$. The deformation is differentiable if the induced map from $G \times M \times I$ into M is. If a deformation φ_t of φ is induced by some differentiable isotopy ψ_t of M (ψ_0 the identity), then φ_t is said to be differentiably trivial. The authors prove that if both G and M are compact, then every differentiable deformation of a differentiable action of G on M is differentiably trivial. Both compactness assumptions are necessary. *S. Smale* (Berkeley, Calif.)

11402:

Rohlin, V. A. Theory of intrinsic homologies. Uspehi Mat. Nauk **14** (1959), no. 4 (88), 3–20. (Russian)

Expository article on cobordism. Basic question: How does one decide whether a given (orientable) manifold is a boundary of some (orientable) manifold-with-boundary? The paper gives a survey of the results, and describes the contributions of various mathematicians (among them the author). The following is essentially a list of section headings: older results (e.g., the characteristic must be even); Stiefel-Whitney and Pontryagin classes (the latter in the author's definition: obstruction to existence of m vector fields, everywhere of rank $\geqq m-1$); Pontryagin's theorem (vanishing of Stiefel-Whitney and Pontryagin numbers); discussion of dimensions 3 and 4; Thom's results on N^n and Ω^n; the author's work on $h: \Omega^* \to N^*$; discussion of the 2-torsion group in Ω^* {but for an error in the last two sections and for the complete description of the ring Ω^* see C. T. C. Wall, #11403 below}. *H. Samelson* (Princeton, N.J.)

11403:

Wall, C. T. C. Determination of the cobordism ring. Ann. of Math. (2) **72** (1960), 292–311.

Let Ω [resp. $\mathfrak{N}$] denote the oriented [resp. non-oriented] cobordism ring (Thom algebra). Thom proved that $\mathfrak{N}$ is a polynomial algebra over Z_2 with one generator x_i in each dimension i not of the form $2^j - 1$. Results of Thom, Milnor, Averbuch show that Ω has no odd torsion and that Ω/T (where T = torsion ideal) is a polynomial ring over Z with one generator Y_{4k} of dimension $4k$ for $k = 1, 2, \cdots$. The author now determines completely the structure of Ω and the natural ring homomorphism $r: \Omega \to \mathfrak{N}$ (which forgets about orientation); some work along the author's lines had already been done by V. A. Rohlin [Dokl. Akad. Nauk SSSR **89** (1953), 789–792; **119** (1958), 876–879; MR **15**, 53; **21** #2238].

The main results are as follows: (1) There exists an exact sequence

$$\to \Omega_q \xrightarrow{2} \Omega_q \xrightarrow{r} \mathfrak{W}_q \xrightarrow{\partial} \Omega_{q-1} \xrightarrow{2} \Omega_{q-1} \to$$

where $\mathfrak{W}_q \subset \mathfrak{N}_q$ consists of cobordism classes which contain a manifold M^q such that $w_1(M^q) \in \operatorname{im}\{H^1(M, \underline{Z}) \to H^1(M, Z_2)\}$. (This is partly contained in Rohlin's work.) $\mathfrak{W} = \sum \mathfrak{W}_q$ is actually a subalgebra of $\mathfrak{N}$. (2) $\mathfrak{W}$ is a polynomial algebra with generators X_{2k-1}, X_{2k} (k not a power of 2) and $X_{2^j}{}^2$, $j = 1, 2, \cdots$. The composite $\bar{\partial}: \mathfrak{W} \xrightarrow{\partial} \Omega \xrightarrow{r} \mathfrak{W}$ is the unique derivation such that $\bar{\partial}(X_{2k}) = X_{2k-1}$, $\bar{\partial}(X_{2k-1}) = 0$, $\bar{\partial}(X_{2^j}{}^2) = 0$. Explicit representative manifolds for the generators are given. (3) $2T = 0$ (i.e., there are no elements of order 4), hence $T = \operatorname{im}(\partial) \overset{r}{\cong} r(T) = \operatorname{im}(\bar{\partial})$. This determines the ring structure of T (and the complete additive structure of Ω). (4) Two oriented manifolds are cobordant if and only if their Pontryagin numbers and Stiefel-Whitney numbers are equal. (5) The generators Y_{4k} for Ω/T can be chosen in such a way that $r(Y_{4k}) = X_{2k}{}^2$. This, together with (3), determines the multiplicative structure of Ω. A description of Ω in terms of generators and relations is given. Other results are: Let $x, y \in \mathfrak{N}$. Then (a) $x^2 \in \operatorname{im}(r)$, (b) $x \in \operatorname{im}(r) \Leftrightarrow$ all Stiefel-Whitney numbers with w_1 as factor vanish on x, (c) $x \notin \operatorname{im}(r)$, $y \in \operatorname{im}(r) \Rightarrow xy \notin \operatorname{im}(r)$.

The proofs are generally not difficult, with the exception of the one for the author's Theorem 1, which has since been circumvented by a considerably simpler argument of M. F. Atiyah [mimeographed, Cambridge, 1960]. *A. Dold* (New York)

11404:

Haefliger, André. Points multiples d'une application et produit cyclique réduit. Amer. J. Math. **83** (1961), 57–70.

Author's introduction: "Le but essentiel de cette note est de déterminer la classe de cohomologie universelle modulo p duale au cycle des points p-uples d'une application f d'une variété V dans une variété M, p étant premier et les points p-uples étant considérés comme des points du produit cyclique de V. Cette classe peut aussi s'interpréter comme une obstruction à trouver dans la classe d'homotopie de f une application sans point p-uple. Elle est en relation étroite avec la classe de plongement Φ_p de Wu [Sci. Sinica **7** (1958), 251–297, 365–387; **8** (1959), 133–150; MR **20** #4825b, 5471; **22** #3000].

"La méthode utilisée donne une détermination explicite de la cohomologie modulo p du p-produit cyclique réduit $V_p{}^*$ d'une variété V. C'est dans la cohomologie de cet espace que se trouvent des obstructions au plongement de V dans une variété M [Wu, op. cit.; et Shapiro, Ann. of Math. (2) **66** (1957), 256–269; MR **19**, 671]. Nous retrouvons les conditions données par Wu pour l'annulation des classes Φ_p lorsque M est l'espace euclidien." *M. W. Hirsch* (Berkeley, Calif.)

11405:

Eells, James, Jr. A class of smooth bundles over a manifold. Pacific J. Math. **10** (1960), 525–538.

The author continues his study of infinite-dimensional manifolds. A differentiable version of the space of paths over B is given, where B is a homogeneous space of a connected (finite-dimensional) Lie group. This provides a differentiable classifying (principal) bundle over $B=G/K$, whose fibre $E(G, K)$ is the group of paths x: $(I, \dot{I}, 0)\to(G, K, e)$ such that x is differentiable for almost all $t\in I$, and the derivative x' is square integrable. An integral formula is given for the characteristic class of this bundle. At this point the reader is supposed to be familiar with the paper by C. B. Allendoerfer and J. Eells in Comment. Math. Helv. **32** (1958), 165–179 [MR **21** #868]. The case $B=S^n$ is considered.
M. A. Kervaire (New York)

11406:

Dolbeault, Pierre. Formes différentielles et cohomologie à coefficients entiers (couples d'Allendoerfer-Eells). Séminaire P. Lelong, 1958/59, exp. 1, 10 pp. Faculté des Sciences de Paris, 1959.

This paper presents an exposition of the results of Allendoerfer and Eells [Comment. Math. Helv. **32** (1958), 165–179; MR **21** #868] concerning the use of pairs of singular differential forms to represent cohomology with integral coefficients. These results are then applied to complex analytic manifolds. Pairs of forms are obtained which represent $H^2(X, Z)$ for a Stein manifold and $H^{1,1}(X, Z)$ for a complex projective space.
C. B. Allendoerfer (Seattle, Wash.)

11407:

★Séminaire d'analyse dirigé par Pierre Lelong. 1re année: 1957/58. Faculté des Sciences de Paris. Secrétariat mathématique, Paris, 1959. ii+99 pp. (mimeographed)

The reports of this seminar are reviewed separately: #11408–11411 below; also #11015, 11024, 11392a.

11408:

Lelong, Piere. Singularités impropres des fonctions holomorphes ret des fonctions plurisousharmoniques. Séminaire P. Lelong, 1957/58, exp. 1, 9 pp. Faculté des Sciences de Paris, 1959.

Der Verfasser betrachtet in Gebieten $G\subset C^n$ (oder allgemeiner in komplexen Mannigfaltigkeiten) abgeschlossene Mengen E, die lokal als Menge der Stellen auftreten, in denen eine subharmonische Funktion $-\infty$ ist. Er gibt an, daß sich beschränkte, in $G-E$ subharmonische Funktionen als solche nach G fortsetzen lassen. Analoge Aussagen werden für plurisubharmonische und holomorphe Funktionen hergeleitet. Es folgen einige Resultate über hinreichende Bedingungen, daß eine stetige Funktion holomorph ist (Untersuchung des Graphen, Verallgemeinerung eines Satzes von Radó). Am Schluß der Note werden Klassen von Teilmengen $A\subset G$ erklärt, derart, daß in bezug auf A die Fortsetzungsaussagen ohne die Beschränktheitsvoraussetzungen gelten.
H. Grauert (Göttingen)

11409:

Dolbeault, Pierre. Espaces analytiques. Séminaire P. Lelong, 1957/58, exp. 4, 13 pp. Faculté des Sciences de Paris, 1959.

In dem Exposé eines Seminarvortrages wird ein Bericht über die wichtigsten Ergebnisse der Theorie der komplexen Räume gegeben. Zunächst werden komplexe Räume nach der Definition von Cartan und Serre eingeführt (β-Räume). Dabei werden beringte Räume verwendet. Sodann wird der Begriff der Normalisierung untersucht. Abschließend definiert der Verfasser analytische Überlagerungen und vergleicht die β-Räume mit dem von H. Behnke und K. Stein angegebenen Begriff des komplexen Raumes.
H. Grauert (Göttingen)

11410:

Norguet, François. Faisceaux et espaces annelés. Séminaire P. Lelong, 1957/58, exp. 10, 12 pp. Faculté des Sciences de Paris, 1959.

In diesem Exposé werden in systematischer Weise allgemeine Begriffe und Sätze aus der Garbentheorie und der Theorie der beringten Räume zusammengetragen, die zum Verständnis des folgenden Exposés [vgl. etwa nachstehendes Referat] erforderlich sind.

Unter einem beringten Raum $(X, \mathfrak{A})$ wird ein topologischer Raum X verstanden, über dem eine Untergarbe $\mathfrak{A}$ von Ringen der Garbe aller komplex-wertigen stetigen Funktionskeime ausgezeichnet ist: $\mathfrak{A}$ umfasse überdies die Garbe aller konstanten Funktionskeime. Garben von $\mathfrak{A}$-Moduln über X werden dann $\mathfrak{A}$-Garben genannt. Sind $(X, \mathfrak{A})$ und $(Y, \mathfrak{B})$ beringte Räume, so ist der Begriff der morphen Abbildung $f: X\to Y$ wohldefiniert. Jeder $\mathfrak{B}$-Garbe $\mathfrak{T}$ über Y ist vermöge f eine $\mathfrak{A}$-Garbe $f^1(\mathfrak{T})$ über X zugeordnet; f^1 ist ein rechtsexakter kovarianter additiver Funktor. Ferner definiert f für jede $\mathfrak{A}$-Garbe $\mathfrak{S}$ über X und jede natürliche Zahl $p\geqq 0$ eine $p$te $\mathfrak{B}$-Bildgarbe $f_{p1}(\mathfrak{S})$ über Y; jedes f_{p1} ist ein kovarianter additiver Funktor; man schreibt f_1 anstelle von f_{01}. Es werden allgemeine Aussagen über die Funktoren f^1, f_{p1} bewiesen.

Man hat für jede $\mathfrak{A}$-Garbe $\mathfrak{S}$ einen kanonischen $\mathfrak{A}$-Homomorphismus $f^1(f_1(\mathfrak{S}))\to\mathfrak{S}$. Ist derselbe surjektiv, so nennt man $\mathfrak{S}$ A-einfach (bzgl. f). Verschwinden alle Garben $f_{p1}(\mathfrak{S})$ für $p\geqq 1$, so heisst $\mathfrak{S}$ B-einfach (bzgl. f). $\mathfrak{S}$ heisst einfach schlechthin, wenn $\mathfrak{S}$ sowohl A-einfach als auch B-einfach ist. Der Begriff der Einfachheit ist transitiv.

Eine $\mathfrak{A}$-Garbe $\mathfrak{S}$ über X heisse eine A-Garbe bzw. eine B-Garbe, wenn $\mathfrak{S}$ A-einfach bzw. B-einfach ist bzgl. der Projektion von X auf einen einzigen Punkt. $\mathfrak{S}$ ist genau dann eine A-Garbe, wenn für jeden Punkt $x\in X$ die Schnittflächen aus $H^0(X, \mathfrak{S})$ über $\mathfrak{A}_x$ den Halm $\mathfrak{S}_x$ erzeugen. $\mathfrak{S}$ ist genau dann eine B-Garbe, wenn $H^p(X, \mathfrak{S})=0$ für alle $p\geqq 1$. Mann nennt $\mathfrak{S}$ eine C-Garbe, wenn $\mathfrak{S}$ sowohl eine A-Garbe als auch eine B-Garbe ist. C-Garben spielen in der Funktionentheorie eine wichtige Rolle; z.B. ist jede kohärente analytische Garbe über einem holomorph-vollständigen komplexen Raum eine C-Garbe.
R. Remmert (Erlangen)

11411:

Norguet, François. Images de faisceaux analytiques cohérents (d'après H. Grauert et R. Remmert). Séminaire P. Lelong, 1957/58, exp. 11, 17 pp. Faculté des Sciences de Paris, 1959.

Es sei Y ein komplexer Raum; mit P_n werde der n-dimensionale komplex-projektive Raum bezeichnet.

$\mathfrak{E}$ sei die Garbe der Keime der holomorphen Schnittflächen im ausgezeichneten Geradenbündel über $Y \times P_n$, das von den Hyperebenen des P_n induziert wird. $\mathfrak{E}^k$ stehe für das k-fache Tensorprodukt von $\mathfrak{E}$ mit sich selbst. Es werden die folgenden Sätze diskutiert und unter Bezugnahme auf Grauert und Remmert, Ann. of Math. (2) **68** (1958), 393–443 [MR **21** #1402] bewiesen. Theorem I_n: Ist $\mathfrak{S}$ eine kohärente analytische Garbe über $Y \times P_n$, so gibt es zu jedem relativ-kompakten Teilbereich Q von Y eine natürliche Zahl $k_0(Q, \mathfrak{S})$, sodass bzgl. der natürlichen Projektion $Q \times P_n \to Q$ alle Garben $\mathfrak{S} \otimes \mathfrak{E}^k | Q \times P_n$ für $k \geq k_0(Q, \mathfrak{S})$ einfach sind. Theorem II_n: Für jede kohärente analytische Garbe $\mathfrak{S}$ über $Y \times P_n$ sind bzgl. der natürlichen Projektion $Y \times P_n \to Y$ alle Bildgarben analytisch und kohärent über Y. Theorem III_n: Ist $\mathfrak{S}$ eine kohärente analytische Garbe über $Y \times P_n$, so gibt es zu jedem relativ-kompakten holomorph-vollständigen Teilbereich Q von Y eine natürliche Zahl $k_0(Q, \mathfrak{S})$, sodass bzgl. der natürlichen Projektion $Q \times P_n \to Q$ alle Garben $\mathfrak{S} \otimes \mathfrak{E}^k | Q \times P_n$ für $k \geq k_0(Q, \mathfrak{S})$ C-Garben sind. (Hinsichtlich der benutzten Begriffe "einfach", "Bildgarbe", "C-Garbe" vgl. vorstehendes Referat.)

Zum Beweise dieser Sätze wird u.a. im Anschluss an H. Cartan gezeigt: Ist $\mathfrak{S}$ eine analytische Untergarbe einer kohärenten analytischen Garbe über einem komplexen Raum X und gilt $H^1(U, \mathfrak{S}) = 0$ für jeden holomorph-vollständigen Teilbereich U von X, so ist $\mathfrak{S}$ kohärent. In Grauert und Remmert, op. cit., wurde diese Aussage nur für den Fall, dass X ein Gebiet im Zahlenraum und $\mathfrak{S}$ eine analytische Untergarbe von $\mathfrak{O}^p$ ist, $p \geq 0$, bewiesen. *R. Remmert* (Erlangen)

11412:

Norguet, François. Dérivées partielles et résidus de formes différentielles sur une variété analytique complexe. Séminaire P. Lelong, 1958/59, exp. 10, 24 pp. Faculté des Sciences de Paris, 1959.

This article is inspired by and is in a sense a completion of some recent work of Leray [Bull. Soc. Math. France **87** (1959), 81–180; C. R. Acad. Sci. Paris **247** (1958), 2253–2257; **248** (1959), 22–28] on residues and Cauchy formulas in several variables; it contains the details of an earlier summary article [C. R. Acad. Sci. Paris **248** (1959), 2057–2059; MR **21** #5195]. Let S be a closed subset of a topological space X such that S and $X-S$ are topological manifolds of dimensions m and n respectively. Then from the coboundary map in the exact cohomology sequence of the pair X, S (cohomology with compact supports, coefficients in a field) and from the Poincaré isomorphism between cohomology and homology (still with compact supports) there follows a homomorphism $\delta: H_{c,q}(S) \to H_{c,q+n-m-1}(X-S)$; the transpose of this map with respect to the duality between homology with compact supports and cohomology with arbitrary closed supports is a homomorphism $r: H^p(X-S) \to H^{p-(n-m-1)}(S)$ called the residue. The aim of the article is the explicit realization of this homomorphism in terms of differential forms.

First suppose that X is a complex analytic manifold, S is a complex analytic submanifold of codimension 1, and s is a holomorphic function on X which vanishes to the first order on S; and consider a cohomology class in $H^p(X-S)$ represented by a C^∞ differential form φ in $X-S$ such that $s^k\varphi$ is C^∞ in X. If $k=1$ there are C^∞ differential forms ψ, θ on X such that $\varphi = (ds/s) \wedge \psi + \theta$ in $X-S$; it follows from the standard Cauchy integral formula that the restriction $\psi|S$, which depends only upon φ, then represents the residue $r(\varphi) \in H^{p-1}(S)$. Actually the function s need only be defined locally; and similar results are discussed for arbitrary k, and even for subvarieties S with singularities. This approach is essentially Leray's formalism of partial differentiation of differential forms; the results were obtained, even for relative cohomology although only for non-singular manifolds S, in the papers of Leray noted above. The first section of this article indeed consists of a survey of this Leray theory. The same result can also be expressed in terms of the Dolbeault formalism [Ann. of Math. (2) **64** (1956), 83–130; MR **18**, 670]. With the notation as above and with $k=1$, let $\tilde{\varphi}$ be the continuation of the differential form φ to a current on X; then $d''\tilde{\varphi} = 2\pi i \tilde{S} \wedge r(\varphi)$, where $r(\varphi)$ is the residue of φ and $\tilde{S}$ is the current of integration over S. Similar results are discussed for arbitrary k.

The Bochner-Martinelli integral formula [Martinelli, Accad. Ital. Mem. Cl. Sci. Fis. Mat. Nat. **9** (1938), 269–283; Comment. Math. Helv. **17** (1945), 201–208; MR **7**, 151; Bochner, Ann. of Math. (2) **44** (1943), 652–673; MR **5**, 116] also leads to a realization of the residue homomorphism in some cases. Let X be the space of n complex variables and S be the hyperplane $z_1 = \cdots = z_p = 0$ $(p = n-m)$, and let $\rho^2 = \sum_{i=1}^p |z_i|^2$ and

$$K = \rho^{-2p} \sum_{i=1}^{p} (-1)^{i-1} \bar{z}_i \, dz_1 \cdots dz_p \, d\bar{z}_1 \cdots \widehat{d\bar{z}_i} \cdots d\bar{z}_p.$$

Then from the Bochner-Martinelli formula it follows that $r(K \wedge \varphi) = \varphi|S$.

The final section of the paper discusses composition of the residue homomorphism corresponding to pairs of subvarieties, and other generalizations.

R. C. Gunning (Princeton, N.J.)

11413:

Borel, A.; Hirzebruch, F. Characteristic classes and homogeneous spaces. III. Amer. J. Math. **82** (1960), 491–504.

In this addendum to their two previous papers on this subject [same J. **80** (1958), 458–538; **81** (1959), 315–382; MR **21** #1586; **22** #988], the authors refine their earlier results in several ways. They first of all free their key integrality theorems of the "ex 2" condition by applying Milnor's result that the Todd genus of an almost complex manifold is always an integer. Then they give general proofs of certain properties of roots of Lie-groups which they had previously verified by checking. Finally they establish simple sufficient conditions for the triviality of the Whitney and Pontryagin classes of a compact manifold, which are directly applicable to certain homogeneous spaces. *R. Bott* (Cambridge, Mass.)

11414:

Holmann, Harald. Quotienten komplexer Räume. Math. Ann. **142** (1960/61), 407–440.

In the case of one complex variable, if a group of automorphisms G operates on a Riemann surface F in a properly discontinuous fashion, then the quotient space F/G has naturally the structure of a Riemann surface. Similar results no longer hold for several variables. When a group of automorphisms G operates on a complex

analytic manifold X, X/G has, in general, non-uniformizable singularities. H. Cartan proved [*Algebraic geometry and topology*, Princeton Univ. Press, Princeton, N.J., 1957; MR **18**, 823; pp. 90–102] that if G operates on a complex space X properly discontinuously, then the quotient X/G has naturally the structure of a complex space. Further, if X is normal, so is X/G.

In the present paper, the author considers those conditions under which the quotient space X/L has naturally the structure of a complex space, where L is a complex Lie group of automorphisms of a complex space X. After stating several definitions and elementary results on complex spaces and the quotient of a ringed space, the author gives the following important result (Satz 12). If a complex Lie group of automorphisms L operates separably on a complex space $(X, \mathfrak{A})$ ($\mathfrak{A}$ being the structural sheaf of X), then the quotient space $(X/L, \mathfrak{A}/L)$ is naturally a complex space. If $(X, \mathfrak{A})$ is normal, so is $(X/L, \mathfrak{A}/L)$. Here, "operate separably" means the following. (a) If two points x' and x'' are not equivalent with respect to L, there exist neighborhoods U' and U'' such that $g(U') \cap U''$ is empty for every $g \in L$. (b) For every point x, there exist a neighborhood U_x, an analytic set s_x in U_x passing through the point x and invariant under the isotropy group L_x, and a neighborhood V_ε of the identity ε in L, such that the natural operation $\Phi(g, x') = g(x')$ maps $V_\varepsilon \times s_x$ onto U_x biholomorphically, and that $g \in L$, $x' \in s_x$, $g(x') \in s_x$ imply $g \in L_x$. (c) For every point x, the isotropy group L_x is finite. Under these assumptions, the canonical projection $\pi: (X, \mathfrak{A}) \to (X/L, \mathfrak{A}/L)$ is a holomorphic mapping. Also a holomorphic L-invariant function f in an open subset U of X is reduced to a holomorphic function f^* on $\pi(U)$, in such a way that $f^* \circ \pi = f$ holds in U.

In contrast to the theorem of H. Cartan, the present condition contains some notions concerning analyticity (e.g., the analytic set s_x). However, the condition of a separably operating group of automorphisms is a purely topological notion. The author proves (Satz 19) that, if a complex Lie group of automorphisms L operates on a complex space X, whose isotropy group L_x is always finite at every point $x \in X$, then a necessary and sufficient condition that L operate separably on X is that L operate on X locally properly. The final condition means that, for every point x, there exists a neighborhood U_x such that the canonical mapping $\Phi: L \times U_x \to X$ is proper (the inverse of a compact set being again compact).

In the final section, several corollaries and applications are discussed. For example, the author gives a condition that X/L be a non-singular manifold when X is a manifold (Satz 24). Further, the normalization of X (Satz 26) and product spaces (Satz 27) are considered.

S. Hitotumatu (Tokyo)

11415:

Ise, Mikio. Some properties of complex analytic vector bundles over compact complex homogeneous spaces. Proc. Japan Acad. **36** (1960), 247–251.

This note gives, without proofs, a number of results on C-manifolds [in the sense of Wang, Amer. J. Math. **76** (1954), 1–32; MR **16**, 518]. Proofs are promised in a future publication. The results of most interest appear to be the following. (1) The connected component of the group of automorphisms of a C-manifold is a reductive complex Lie group. (2) The tangent vector bundle of an irreducible Kählerian C-manifold is indecomposable.

M. F. Atiyah (Oxford)

AUTHOR INDEX

PART A